Const.	$f(x) = \#$	0
Linear	$f(x) = 3x+1$	1
Quad.	$f(x) = 2x^2+3x+1$	2

2.1 Linear and Quadratic functions and Modeling

- Polynomial Function – Function in the form
  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$
  such that $a_n \neq 0$ n = non-neg integer

(Type of poly)
• zero function $f(x)=0$ undef. degree & no leading coefficient

Examples

Which of the following are polynomials? If they are polynomials also state the degree and leading coefficient.

a) $f(x)=7x^{-3}-6$ No b/c -3 exp.
b) $f(x)=-9+4x^2$ yes coeff 4 degree 2
c) $f(x)=\sqrt{x-3}$ No b/c can't be simp into correct form
d) $f(x)=(x^2)^3-7x+2$ Yes, coeff. 1, degree 6
x^6-7x+2

Example

Write an equation for the linear function f such that f(3) = 3 and f(5) = 8. Support graphically and numerically.

Slope = Rate of change

(3, 3) (5, 8)

$\frac{8-3}{5-3} = \frac{5}{2}$

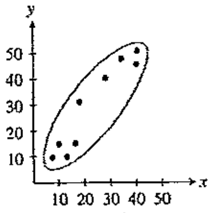
$y-3 = \frac{5}{2}(x-3)$

$y = \frac{5}{2}x - \frac{9}{2}$

→ Check graph & Plug into confirm numerically

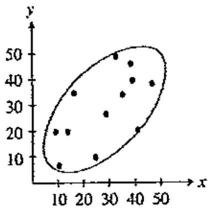
- Linear correlation

Narrow oval = Strong correlation



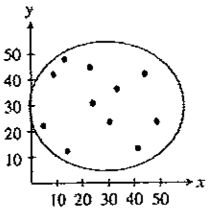
Strong positive linear correlation

(a)



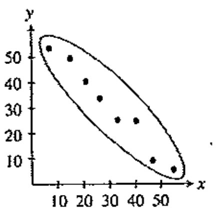
Weak positive linear correlation

(b)



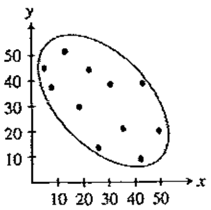
Little or no linear correlation

(c)



Strong negative linear correlation

(d)



Weak negative linear correlation

(e)

$y = -2x^2 + 8x + 8$

$-2(x^2 - 4x - 4)$

$-2(x^2 - 4x + 4) - 4 - 4)$

$-2(x - 2)^2 + 16$

always pos

always neg

$y = 16$

$x = 2$

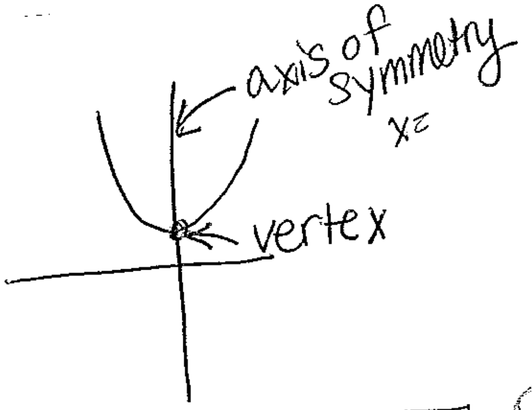
$x - 2 = 0$

Properties of t

1. -1 ≤ r ≤ 1
2. When r > 0,
3. When r < 0,
4. When $|r| \approx$
5. When $r \approx 0$

*** Note that correlati

Day 2 – Quadratics



- *Standard Form* $ax^2 + bx + C$

- *Vertex Form* $a(x-h)^2 + K$

$a\ (x-h)(x-h)$

$a(x^2 - 2xh - h^2) + K$

$ax^2 - 2ahx - ah^2 + K$

$= ax^2 (-2ah)x - (ah^2 + K)$

~~Vertex~~ Axis $\boxed{x = \frac{-b}{2a}}$

Vertex (h, K) $\boxed{K = c - ah^2}$

Axis $x = h$

↑ don't memorize just plug in x

Example

Use the vertex form of a quadratic function to find the vertex and axis of the graph of $f(x) = -2x + x^2 - 3$. Rewrite the equation in vertex form.

$f(x) = -2x + x^2 - 3$

$= x^2 - 2x - 3$

$a = 1 \quad c = -3$

$b = -2$

$h = \frac{-b}{2a}$

$= \frac{2}{2 \cdot 1} = ①$

$K = f(1)$

$1^2 - 2 \cdot 1 - 3$

$1 - 2 - 3$

$\textcircled{-4}$

$a(x-1)^2 + -4$

$\boxed{1(x-1)^2 + -4}$

Use the vertex form of a quadratic function to find the vertex and axis of the graph of f(x) = 2-6x+x^2. Rewrite the equation in vertex form.

$f(x) = x^2 - 6x + 2$

$h = \frac{-b}{2a}$

$h = \frac{6}{2 \cdot 1} = 3$

$K = 3^2 - 6 \cdot 3 + 2$

$9 - 18 + 2$

-7

$(3, -7)$

$1(x-3)^2 + -7$

Example

Use completing the square to describe the graph of $f(x) = 2x^2 + 4x - 1$. Support your answer graphically.

$f(x) = 2x^2 + 4x - 1$ $\quad \frac{2}{2} = 1^2$

$2(x^2 + 2x \quad) - 1$

$2(x^2 + 2x + 1 - 1) - 1$

$2(x^2 + 2x + 1) - 2 - 1$

$\boxed{2(x+1)^2 - 3}$ Vertex $(-1, -3)$

→ opens Up
Axis of S.
$x = -1$

➤ ***Vertical Free-Fall Motion***

$s(t) = -\frac{1}{2} g t^2 + v_0 t + s_0$

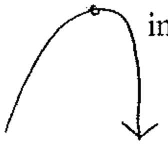
height

and $v(t) = -gt + v_0$

$g = 32\,\text{ft/sec}^2 \approx 9.8\,\text{m/sec}^2$

Example #61

As a promotion for the Houston Astros downtown ballpark, a competition is held to see who can throw a baseball the highest from the front row of the upperdeck of seats, 83 feet above field level. The winner throws the ball with an initial vertical velocity of 92 ft/sec and it lands on the infield grass. $s(t) = -16t^2 + 92t + 83$



a) Find the maximum height of the baseball. Vertex $(2.87, 215.25)$

b) How much time is the ball in the air? zero ≈ 6.5 sec

c) Determine its vertical velocity when it hits the ground. $\boxed{\approx 116 \text{ ft/sec}}$ [illegible]

a) $s(t) = $ ~~$-\frac{1}{2} \cdot 32t^2 + 92t + 83$~~

~~$= -16t^2 + 92t + 83$~~

~~$= -4(4t^2 + 23t + 132.25 - 132.25) + 83$~~

~~$= -4(4t^2 + 11.5)^2 + 529 + 83$~~

~~$-4(4t^2 + 11.5)^2 + 612$~~

OR $\frac{-b}{2a}$ $\frac{-92}{2 \cdot 16}$ $= ($~~-2.875~~$,$ ~~49.75~~$)$

b) $0 = -16t^2 + 92t + 83$

$$\frac{-92 \pm \sqrt{92^2 - 4 \cdot -16 \cdot 83}}{2 \cdot -16}$$

6.5

c) [illegible]

$V(6.5) = -\overset{32}{g} \cdot 6.5 + 92$

$\sqrt[3]{x}, \frac{1}{x^2}$ $X, X^2, X^3, X^{-1}, X^{1/2}$

$\frac{1}{x}$ $\sqrt{x}$

2.2 Power Functions with Modeling

- Power Function = $K \cdot x^a$ where K & a are nonzero constants

power

constant of variation

- Direct Variation

$C = 2\pi r$ (Pos Powers)

- Inverse Variation

$V = K/p$ (Neg Powers)

Example

Determine whether the function is a power function given that c, g, k , and pi, represent constants. For those that are power functions, state the power and constant of variation for the function, graph it, and analyze it.

a) $f(x) = \sqrt[4]{x} = x^{1/4}$ Power 1/4 constant 1, $[0,\infty)$ R $[0,\infty)$ Inc cont

b) $g(x) = \frac{1}{x^3}$ $1 \cdot x^{-3}$ Power -3 Const 1, D: $(0,\infty) \cup (-\infty,0)$ R $(-\infty,0) \cup (0,\infty)$

c) $f(x) = -.5x^5$ Power 5 Constant $-.5$

d) $h(x) = 3 \cdot 2^x$ Not a Power Function

e) $E(m) = mc^2$ Power 1 constant c^2

f) $I = \frac{k}{d^2}$ Power -2, Const. K

$= Kd^{-2}$

E = mass $\cdot$ (Speed of lt$)^2$

Example

Describe how to obtain the graph of the given function from the graph of $g(x) = x^n$ with the same power n. Sketch the graph by hand and support your answer with a grapher.

a) $f(x) = 3x^3$ Vert Stretch 3

b) $f(x) = -0.5x^4$ Vert Shrink 1/2 & reflect over x-axis

c) $f(x) = -2x^5$ Vert stretch 2 & reflect over x-axis

d) $f(x) = .5x^6$ Vert Shrink 1/2

o Monomial $f(x) = K$ or $f(x) = K \cdot x^n$ ← pos int.

constant

Graphs of Power Functions $f(x) = K \circ x^{a}$

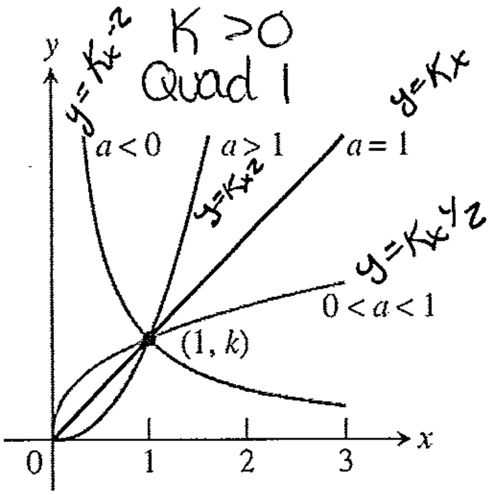
if $\cancel{X}=1$, then goes through $(1, K)$

When $x<0$

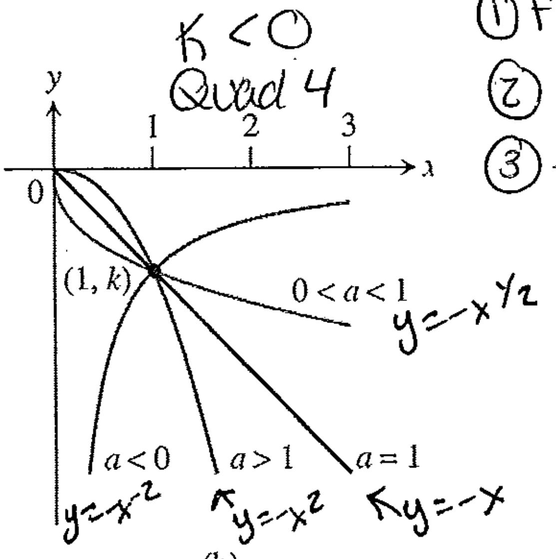
① f undefined $x^{1/2}$ $\sqrt{x}$

② f is even x^{-2}

③ f is odd (origin) x^{-1}



(a)



(b)

Example – Graphing Power Functions

State the values of the constants *k* and *a*. Describe the portion of the curve that lies in Quadrant I or IV. Determine whether *f* is even, odd, or undefined for $x < 0$. Describe the rest of the curve if any. Graph the function to see whether it matches the description.

a) $f(x) = 3x^{-3}$ $K=3$ • $(1,3)$ • odd-origin • dec
$a=-3$ • Asymptotic to both axes $3(-x)^{-3} = \frac{3}{(-x)^3} = -\frac{3}{x^3} = -3x^{-3} = -f(x)$

b) $f(x) = -x^{1.5}$ $K=-1$ • $(0,0)$ through $(1,-1)$
$a=1.5>1$ • 4th quad dec, undefined at zero $(-\sqrt{x})^3$

c) $f(x) = -x^{0.8}$ $K=-1$ $0<a<1$, contains $(0,0)$, through $(1,-1)$. • 4th quad decreasing

even $f(-x) = -(-x)^{.8}$

$= -(-x)^{4/5}$

$= -\sqrt[5]{-x}^{\,4}$

$= -(-\sqrt[5]{x})^4$

$= -(\sqrt[5]{x})^4$

$= -x^{.8} = f(x)$

(Day 2)

Note involves logs so no zeros

Example

Use the following data to obtain a power function from speed p versus distance traveled d. Then use the model to predict the speed of the ball at impact given that impact occurs when d=1.8 m.

Distance (m)	Speed (m/s)
.00000	.00000
.04298	.82372
.16119	1.71163
.35148	2.45860
.59394	3.05209
.89187	3.74200
1.25557	4.49558

$4.027x^{.494}$

1.8

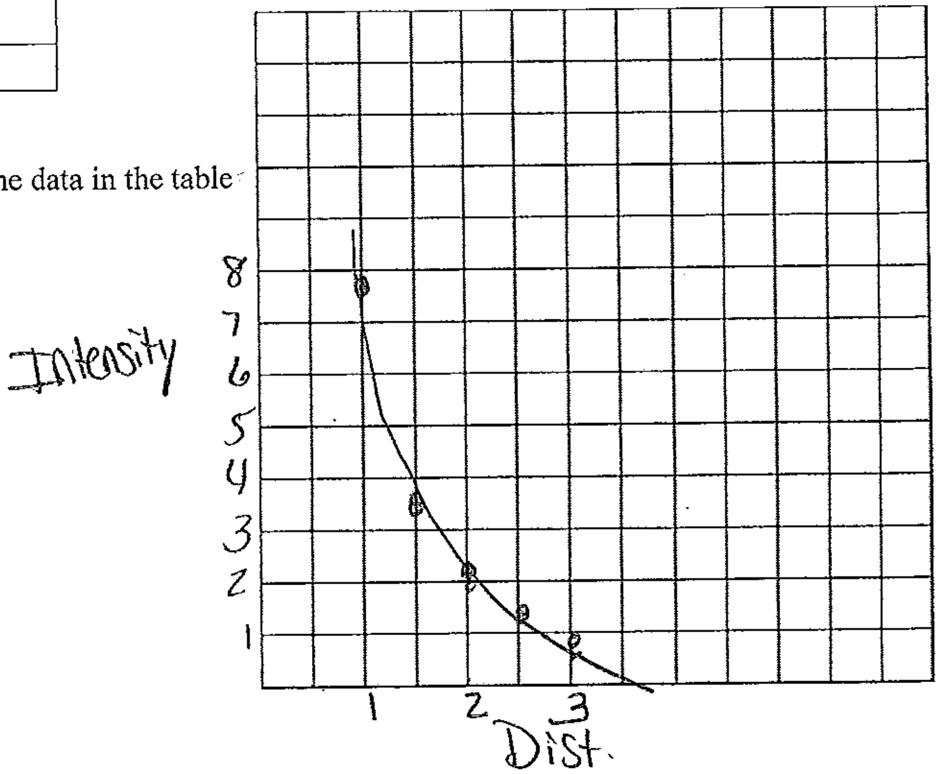
≈ 5.4 m/sec

Example

#57 Velma and Reggie gathered the data in the table below using a 100 watt lightbulb and a CBL with a light intensity probe.

Distance (m)	Intensity (W/m^2
1	7.95
1.5	3.53
2	2.01
2.5	1.27
3	.9

a) Draw a scatterplot of the data in the table



b) Find the power regression model. Is the power close to the theoretical value of a= -2?

$7.93x^{-1.987}$ Yes

c) Superimpose the regression curve on the scatter plot.

d) Use the regression model to predict the light intensity at distances of 1.7m and 3.4m.

$7.93 \cdot 1.7^{-1.987}$ $7.93 \cdot 3.4^{-1.987}$

① 2.76 W/m^2 ② .697 W/m^2

2.3 Polynomial Functions of Higher Degree with Modeling

All poly
Contin
·Smooth
curves

- Cubic Function (poly of deg. 3)

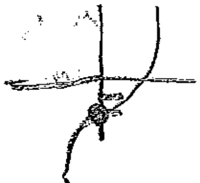
- Quartic Function (poly of deg. 4)

Example

Describe how to transform the graph of an appropriate monomial function $f(x) = a_n x^n$ into the graph of the given function. Sketch the transformed graph by hand and support your answer with a grapher. Compute the location of the y-intercept as a check on the transformed graph.

a) $f(x) = 2(x-1)^3$ (0, −2) y-int.

Shift $2x^3$ 1 Rt



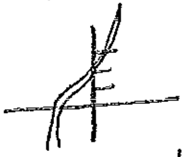
b) $f(x) = -(x+1)^4 - 2$

Shift $-x^4$ down 2
1 left

(0, −3)

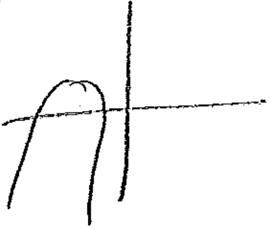
c) $f(x) = 3(x+1)^3$

Shift $3x^3$ 1 left
(0, 3)



d) $f(x) = -(x+2)^4 + 2$

Shift $-x^4$
2 left, up 2 (0, −14)



Example

Graph the polynomial function, locate its extrema and zeros and explain how it is related to the monomials from which it is built. $f(x) = x^3 + x$

$x(x^2+1)$

Inc $(-\infty, \infty)$

No extrema

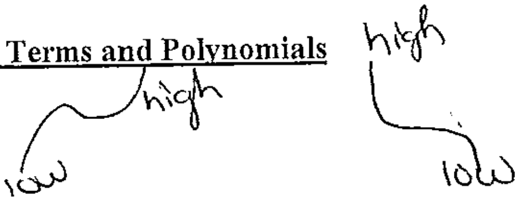
$x(x^2+1)$

zero is $x=0$

- odd like beg

- looks like x^3 but flat near orig. like x

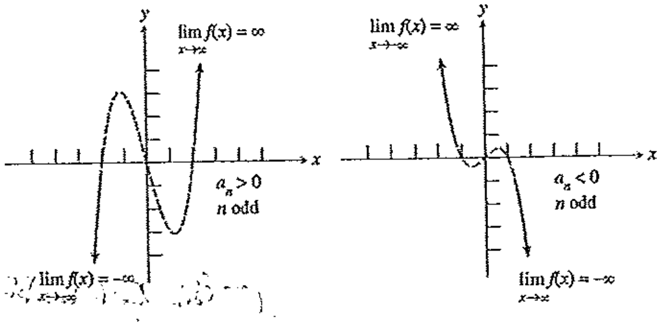
Leading Terms and Polynomials



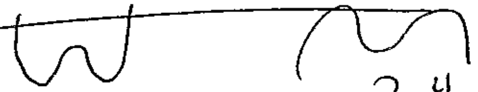
Cubics

$y = 3x^3$ $y = -3x^3$

$y = 7x^3$ $y = -7x^3$

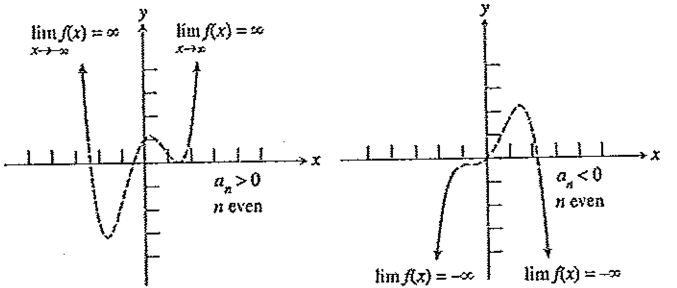


Quartic



$y = 3x^4$ $y = -3x^4$

$y = 7x^4$ $y = -7x^4$



*We know turns = deg - 1

so at most n-1 local extrema

and at most n zeros

Examples

Graph the functions in a viewing window that shows all of its extrema and x-intercepts.
Describe the end behavior using limits.

[-5, 3]
[-8, 3]

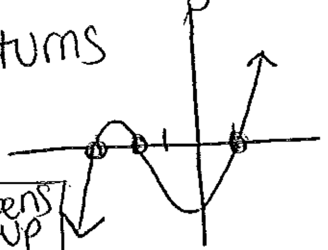
a) $f(x) = (x-1)(x+2)(x+3)$

$\lim_{x\to\infty} = \infty$

1 in front low to high

cubic = 2 turns

$\lim_{x\to-\infty} = -\infty$



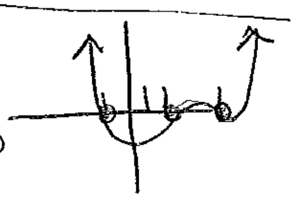
[-5, 5]
[-14, 6]

b) $f(x) = (x-2)^2(x+1)(x-3)$

$\lim_{x\to\infty} f(x) = \infty$

- opens up
- Quartic = 3 turns
- 2, -1, 3

$\lim_{x\to-\infty} f(x) = \infty$



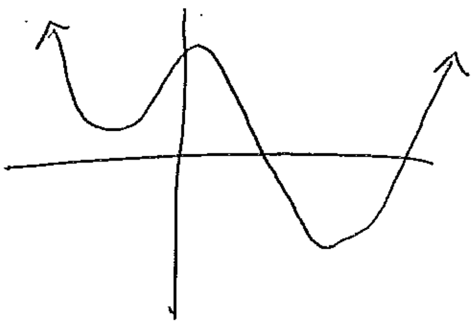
[-3, 5]
[-50, 50]

c) $f(x) = 2x^4 - 5x^3 - 17x^2 + 14x + 41$

- Quartic = 3 turns
- 2 in front opens up
-

$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = +\infty$



(Day 2)

Remember the Zeros of the functions represent the x-intercepts graphically

$$f(x) = (x-4)^2(x+3)^4$$

- Repeated Zero —
- Multiplicity — ~~exponent~~ # of times factor is in polynomial

o even mult. not cross x-axis

o odd does at (c, 0)

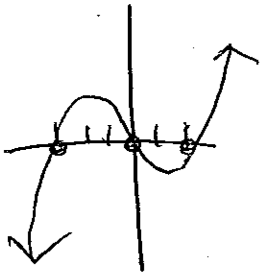
Examples

Find the zeros of $f(x) = x^3 + x^2 - 6x$

$x(x^2+x-6)$

$x(x+3)(x-2)$

zeros 0, -3, 2

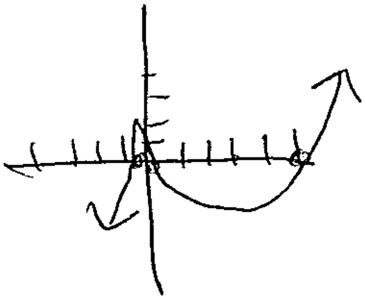


Find the zeros of $f(x) = x^3 - 9x^2 - 10x$

$x(x^2-9x-10)$

$x(x-10)(x+1)$

Zeros 0, 10, -1



- **Zeros of Odd and Even Multiplicity** - If a polynomial function ~~f~~ has a real zero c of odd multiplicity, then the graph of f crosses the x-axis at $(c, 0)$ and the value of f changes sign at $x = c$. If a polynomial function f has a real zero c of even multiplicity, then the graph of f does not cross the x-axis at $(c, 0)$ and the value of f does not change sign at $x = c$.

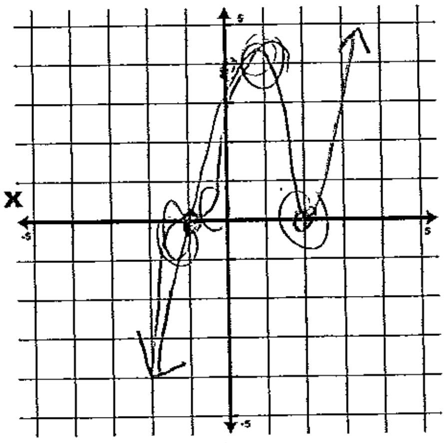
o Even 2 4 6 (squares, so sign doesn't chng)

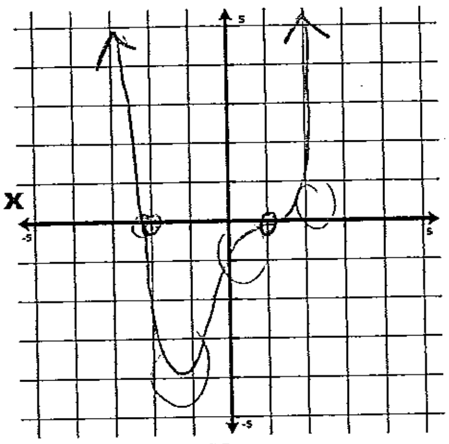
"Kisses axis"

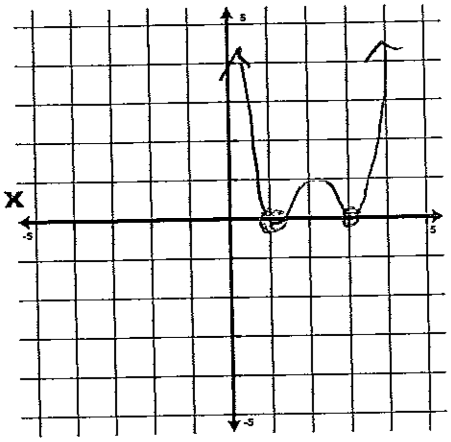
o

Examples

State the degree and list the zeros of each function. State the multiplicity of each zero and whether the graph crosses the x-axis at the corresponding x-intercept. Then sketch the graph of f by hand.







$f(x)=(x+1)^3(x-2)^2$

Y-int (0,4)

Deg 5

zeros $-1, 2$

↑ Yes odd

↑ No even

$f(x)=(x+2)(x-1)^3$

Deg 4

zeros $-2, 1$

Yes crosses

$f(x)=(x-3)^2(x-1)^2$

Deg 4

zeros 3, 1

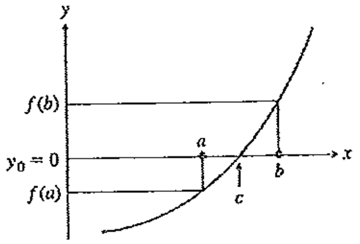
not cross even

INTERMEDIATE VALUE THEOREM

If a and b are real numbers with $a < b$ and if f is continuous on the interval $[a,b]$, then f takes on every value between $f(a)$ and $f(b)$.

In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0=f(c)$ for some number c in $[a,b]$.

In particular, if $f(a)$ and $f(b)$ have opposite signs (i.e., one is negative and the other is positive, then $f(c) = 0$ for some number c in $[a, b]$.



Example [-5,5] [-.4,.1] [-.01,.01]

Find all the real zeros of $f(x)= x^4-1.43x^3-1.09x^2-.214x-.012$

- At most 4 zeros

$x \approx 2$ $x= -.2, -.3, -.1$

2.4 Real Zeros of Polynomial Functions

Simplify using long division.

$\frac{8150}{25}$ Divident Divisor

326 Quotient

$25\,\overline{)\,8150}$

-75

65

-50

150

-150

0

$\frac{3x^3+5x^2+8x+7}{3x+2}$

x^2+x+2

$3x+2\,\overline{)\,3x^3+5x^2+8x+7}$

$-\;3x^3+2x^2$

$3x^2+8x$ R3

$-3x^2+2x$

$6x+7$

$6x+4$

3

$(x^2+x+2)(3x+2)+3=$

- Recall in division: (divisor) (quotient) + (remainder) = dividend

Example

Divide f(x) by d(x) and write a summary statement in polynomial form and fraction form.

$f(x) = x^3 + 4x^2 + 7x - 9 \qquad d(x) = x + 3$

x^2+x+4 R-21

$x+3\,\overline{)\,x^3+4x^2+7x-9}$

x^3+3x^2

x^2+7x

x^2+3x

$4x-9$

$-4x+12$

-21

$(x+3)(x^2+x+4)-21=x^3+4x^2+7x-9$

$\frac{x^3+4x^2+7x-9}{x+3}=x^2+x+4+\frac{-21}{x+3}$

$p(K) = (x-K)\,q(K) + r(K)$ $x-K$ is the zero $x = K$

$(K-K)$

$0\,q(K) + r(K)$

- **Remainder Theorem** – If a polynomial f(x) is divided by x-k, then the remainder is $r = P(K)$
- **Factor Theorem** – A polynomial function f(x) has a factor of x – k iff f(k)=0
- **Fundamental Connections for Polynomial Functions**
  - x = k is a solution of the equation f(x) = 0
  - k is a zero of the function f
  - K is an x-intercept of the graph of y = f(x)
  - X-k is a factor of f(x)

Example

Find the remainder when $f(x) = 2x^2 - 4x + 2$ is divided by:

Use remainder thm.

a) X-3 $2\cdot3^2 - 4\cdot3 + 2 = \boxed{8}$

b) X+3 $2\cdot-3^2 - 4\cdot-3 + 2 = 32$

c) X-1 $2\cdot1^2 - 4\cdot1 + 2 = 0$

If it is a factor, then remainder = zero

Chk

$$\begin{array}{r} 2x+2 \\ x-3\,\overline{)\,2x^2-4x+2} \\ -\ 2x^2-6x\ \downarrow \\ \hline 2x+2 \\ -\ 2x-6 \\ \hline 8 \end{array}$$

Examples

Find the remainder when $f(x) = 2x^2 - 3x + 1$ is divided by x – 2.

$2\cdot2^2 - 3\cdot2 + 1$

$8 - 6 + 1 = \boxed{3}$

Find the remainder when $f(x) = 2x^3 - 3x^2 + 4x - 7$ is divided by x – 2.

$2\cdot2^3 - 3\cdot2^2 + 4\cdot2 - 7$

$16 - 12 + 8 - 7$

$\boxed{5}$

- **Synthetic Division** "Short cut" w/ linear divisors

①

$$\begin{array}{r|l} & 2x^2-3x+4 \\ \hline x-3 & 2x^3-3x^2-5x-12 \\ & 2x^3-6x^2 \downarrow \\ \hline & 3x^2-5x \\ & 3x^2-9x \downarrow \\ \hline & [illegible] \\ & 4x-12 \\ & 4x-12 \\ \hline & 0 \end{array}$$

Ex

$$x-2 \,\overline{\smash{)}\,2x^2-3x+1}$$

$$\begin{array}{r|rrr} 2 & 2 & -3 & 1 \\ & \downarrow & 4 & 2 \\ \hline & 2 & 1 & 3 \end{array}$$

$2x+1$ R 3

Example

Divide $2x^3 - 3x^2 - 5x - 12$ by x – 3 using synthetic division and write a summary statement in fraction form.

$$\frac{2x^3-3x^2-5x-12}{x-3}$$

$$\begin{array}{r|rrrr} 3 & 2 & -3 & -5 & -12 \\ + & \downarrow & 6 & 9 & 12 \\ \hline & 2 & 3 & 4 & \boxed{0} \\ & \downarrow & \downarrow & \downarrow & \\ & x^2 & x & c & \end{array}$$

Opp b/c we want x = 0

Remainder

$$= \boxed{2x^2+3x+4}$$

$$\frac{x^3-3x^2-4x+12}{x+2}$$

$$\begin{array}{r|rrrr} -2 & 1 & -3 & -4 & 12 \\ + & \downarrow & -2 & 10 & -12 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\boxed{x^2-5x+6}$$

(Day 2) - Rational Zeros Theorem

$f(x) = a_n x^n + \dots + a_0 \qquad a_0 \neq 0$

If $x = p/q$ is a rational zero of f, where p & q have no common int. factors other than ±1 then

- p is int. factor of a_0
- q is factor of a_n

Example

Find the rational zeros of $f(x) = x^3 - 3x^2 + 1$

Factors 1: ±1 ±1
Factors 1: ±1

$f(1) = 1^3 - 3 \cdot 1^2 + 1 = -1 \neq 0$

$f(-1) = (-1)^3 - 3(-1)^2 + 1 = -3 \neq 0$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 0 & 1 \\ & \downarrow & 1 & -2 & -2 \\ \hline & 1 & -2 & -2 & -1 \end{array}$$

Remainder ↗

- No rational zeros
- 3 irrat. zeros

Example

Find the rational zeros of $f(x) = 2x^3 + x^2 - 2x - 1$

Factors of -1: 1, -1
Factors of 2: -1, 1, 2, -2 : ±1, ±1/2

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -2 & -1 \\ & \downarrow & 2 & 3 & 1 \\ \hline & 2 & 3 & 1 & 0 \end{array}$$

$2x^2 + 3x + 1$ ↑ remaind.

$(x-1)(2x^2+3x+1)$

$(x-1)(2x+1)(x+1)$ $\boxed{1, -\frac{1}{2}, -1}$

Upper and Lower Bounds for Real Zeros

f(x) divided by x − K

- If K ≥ 0, last line every # nonneg, then K is an upperbound
- If K ≤ 0, alt. nonneg & nonpos., then k is lowerbound

Example

Prove that all of the real zeros of $f(x) = 2x^4 - x^3 - 7x^2 + 3x + 3$ must lie in the interval [-5
5].

$$\begin{array}{r|rrrrr} 5 & 2 & -1 & -7 & 3 & 3 \\ & \downarrow & 10 & 45 & 190 & 965 \\ \hline & 2 & 9 & 38 & 193 & 969 \end{array}$$

← upperbound all pos.

$$\begin{array}{r|rrrrr} -5 & 2 & -1 & -7 & 3 & 3 \\ & \downarrow & -10 & 55 & -240 & 1185 \\ \hline & 2 & -11 & 48 & -237 & 1188 \end{array}$$

← lower alt.

Now Find all of the real zeros of $f(x) = 2x^4 - x^3 - 7x^2 + 3x + 3$.

$$\frac{\text{Factors of } 3}{\text{Factors of } 2} : \frac{\pm 1 \quad \pm 3}{\pm 1 \quad \pm 2} : \boxed{\pm 1}, \boxed{\pm \tfrac{1}{2}}, \pm \tfrac{3}{2}, \pm 3$$

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -7 & 3 & 3 \\ & \downarrow & 2 & 1 & -6 & -3 \\ \hline & 2 & 1 & -6 & -3 & 0 \end{array}$$

$(2x^3 + x^2 - 6x - 3)(x-1)$

$$\begin{array}{r|rrrr} -\tfrac{1}{2} & 2 & 1 & -6 & -3 \\ & \downarrow & -1 & 0 & 3 \\ \hline & 2 & 0 & -6 & 0 \end{array}$$

$(x-1)(x+\tfrac{1}{2})(2x^2-6)$

$(x-1)(x+\tfrac{1}{2})2(x^2-3)$

$2(x-1)(x+\tfrac{1}{2})(x-\sqrt{3})(x+\sqrt{3}$

$1, -\tfrac{1}{2}, \sqrt{3}, -\sqrt{3}$

Example

Find all of the real zeros of the function, finding the values whenever possible. Identify each zero as rational or irrational. $f(x) = x^4 - 3x^3 - 6x^2 + 6x + 8$

~~Factors of 1 : ±1~~

~~Factors of 8 : ±1, ±8, ±2, ±4~~

~~$\boxed{\pm 1}, \pm \tfrac{1}{8}, \pm \tfrac{1}{2}, \pm \tfrac{1}{4}$~~

$$\frac{\text{Factors of } 8}{\text{Factors of } 1} : \frac{\pm 1, \pm 8, \pm 2, \pm 4}{\pm 1}$$

$\boxed{\pm 1}, \pm 8, \pm 2, \boxed{\pm 4}$

$$\begin{array}{r|rrrrr} -1 & 1 & -3 & -6 & 6 & 8 \\ & \downarrow & -1 & 4 & 2 & -8 \\ \hline & 1 & -4 & -2 & 8 & 0 \end{array}$$

$(x+1)(x^3 - 4x^2 - 2x + 8)$

$$\begin{array}{r|rrrr} 4 & 1 & -4 & -2 & 8 \\ & \downarrow & 4 & 0 & -8 \\ \hline & 1 & 0 & -2 & 0 \end{array}$$

$(x+1)(x^2-2)(x-4)$

$-1, 4, \pm\sqrt{2}$

2.5 Complex Zeros and the Fundamental Theorem of Algebra

- **Fundamental Theorem of Algebra** – A polynomial of deg. n has n complex zeros (real & non real). (some may be repeated)

- **Linear Factorization Theorem** – If f(x) is a poly of deg $n>0$, then f(x) has n linear factors and $f(x)=a(x-z_1)(x-z_2)\cdots$ z_1, z_2 are compl. zeros

Examples

Write the polynomial function in standard form, and identify the zeros of the function and the x-intercepts of its graph.

a) F(x) = (x - 2i)(x + 2i) $x^2 + \cancel{2xi} - \cancel{2xi} - 4i^2$ $\boxed{x^2+4}$ $i^2 = -1$ $x^2=-4$

Zeros 2i & -2i
No x-int

b) F(X) = (x – 5) (x - √2i) (x +√2i)

$x^2 - 2i^2$

$(x-5)(x^2+2)$ -1

~~$x^3+2x-5x^2+-10$~~

$x^3-5x^2+(2x-10)$

~~2(x+5)~~

5, $\sqrt{2}i$, $-\sqrt{2}i$

#1-4

- **Complex Conjugate Zeros Theorem** – f(x) is a polynomial w/ real coefficients If a & b are real #s w/ $b \neq 0$, and $a+bi$ is a zero, then complex conjugate $a-bi$ is also a zero

Example

#5-12

Write a polynomial function of minimum degree in standard form with real coefficients whose zeros include –2, 1, and 3 + *i*. → implies $3-i$ is a zero

real so

so $x-(3-i)$ $x-(3+i)$

$(x+2)(x-1)$

$x^2-(3+\cancel{i})x-(3-\cancel{i})x+9$

$(x^2+x-2)(x^2-6x+10) = x^4-5x^3+2x^2+22x-20$

Write a polynomial function of minimum degree in standard form with real coefficients whose zeros and their multiplicities include those listed 1 (multiplicity 2), -2 (multiplicity 3).

13-16

$(x-1)^2(x+2)^3$

$(x^2-2x+1)(x^2+4x+4)(x+2)$

$x^3+2x^2+4x^2+8x+4x+8$

$(x^2-2x+1)(x^3+6x^2+12x+8)$

$(\cancel{x^5}+\cancel{6x^4}+12x^3+8x^2-\cancel{2x^4}$
$-\cancel{12x^3}-24x^2-16x+$
$\cancel{x^3}+6x^2+12x+8)$

$x^5+4x^4+x^3+-10x^2-4x+8$

Example

Find all zeros of $f(x) = x^5 - 3x^3 + 6x^2 - 28x + 24$, and write $f(x)$ in its linear factorization.

$\frac{24}{1}$

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & -3 & 6 & -28 & 24 \\ & \downarrow & 1 & 1 & -2 & 4 & -24 \\ \hline & 1 & 1 & -2 & 4 & -24 & 0 \end{array}$$

$(x-1)(x^4+x^3-2x^2+4x-24)$

$$\begin{array}{r|rrrrr} 2 & 1 & 1 & -2 & 4 & -24 \\ & \downarrow & 2 & 6 & 8 & 24 \\ \hline & 1 & 3 & 4 & 12 & 0 \end{array}$$

$(x-1)(x-2)(x^3+3x^2+4x+12)$

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 4 & 12 \\ & \downarrow & -3 & 0 & -12 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

$(x-1)(x-2)(x+3)(x^2+4)$

$\boxed{(x-1)(x-2)(x+3)(x+2i)(x-2i)}$

$4(x-(1-2i))(x-(1+2i))(x-(-1+\frac{3}{2}i))(x-(-1-\frac{3}{2}i))$

Example

The complex number z = 1-2i is a zero of $f(x) = 4x^4 + 17x^2 + 14x + 65$. Find the remaining zeros of f(x) and write in its linear factorization.

$$\begin{array}{r|rrrrr} 1-2i & 4 & 0 & 17 & 14 & 65 \\ & \downarrow & 4-8i & -12-16i & -27-26i & -65 \\ \hline & 4 & 4-8i & 5-16i & -13-26i & 0 \end{array}$$

$(4-8i)(1-2i) = 4-8i-8i+\underset{-1}{16i^2}$

$(1-2i)(5-16i)$

$5-16i-10i+\underset{-1}{32i^2}$

$-27-26i$

$(1-2i)(-13-26i)$

$-13-26i+26i+\underset{-1}{52i^2}$

-65

So $1-2i$ & $1+2i$ are zeros.

$$\begin{array}{r|rrrr} 1+2i & 4 & 4-8i & 5-16i & -13-26i \\ & \downarrow & 4+8i & 8+16i & 13+26i \\ \hline & 4 & 8 & 13 & 0 \end{array}$$

$4x^2+8x+13$

Use quad

$-1 \pm \frac{3}{2}i$

Example

Write $f(x) = x^5 - x^4 - 2x^3 + 2x^2 - 3x + 3$as a product of linear and irreducible quadratic factors, each with real coefficients.

Fact: ±3, ±1

Fact: ±1

$$\begin{array}{r|rrrrrr} 1 & 1 & -1 & -2 & 2 & -3 & 3 \\ & \downarrow & 1 & 0 & -2 & 0 & -3 \\ \hline & 1 & 0 & -2 & 0 & -3 & 0 \end{array}$$

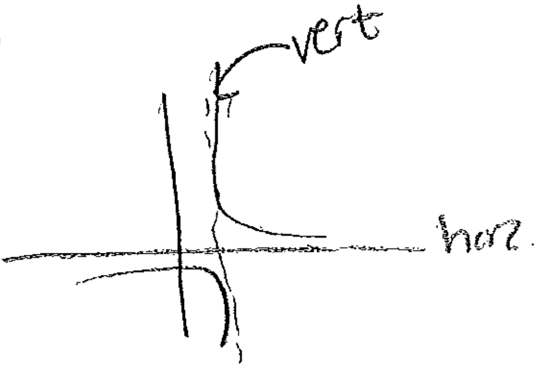
$(x-1)(x^4-2x^2-3)$

$(x-1)(x^2-3)(x^2+1)$

$(x-1)(x-\sqrt{3})(x+\sqrt{3})(x^2+1)$

complex

$\lim_{x \to 2^-} g(x) = -\infty$ $\lim_{x \to -\infty} g(x) = 0$



2.6 Graphs of Rational Functions

- **Recall:**
  - **Horizontal Asymptotes**

$y = b$ if $\lim_{x \to \pm\infty} f(x) = b$

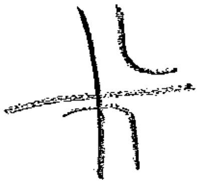
  - **Vertical Asymptotes**

(Real zeros of denom) ← not zeros of num.

$\lim_{x \to a^-, a^+} f(x) = \pm\infty$ $x = a$

Example

Describe how the graph of the given function can be obtained by transforming the graph of the reciprocal function f (x) = 1/x . Identify the horizontal and vertical asymptotes and use limits to describe the corresponding behavior. Sketch the graph of the function.



a) $g(x) = \frac{3}{x-2}$

$g(x) = 3\left(\frac{1}{x-2}\right)$

$= 3f(x-2)$

2 right, vert stretch 3

$x=2$ vert asym. $y=0$ hor. asym.

b) $h(x) = \frac{2x+1}{x+2}$

$$\begin{array}{r} 2 \\ x+2 \overline{) 2x+1} \\ \underline{2x+4} \\ -3 \end{array}$$

$h(x) = 2 - \frac{3}{x+2}$

$= -3f(x+2) + 2$

2 left, Vert Stretch 3, Up 2, reflect over x-axis

$x = -2$, $y = 2$

$\lim_{x \to -2^-} h(x) = \infty$

$\lim_{x \to -2^+} h(x) = -\infty$

$\lim_{x \to \infty} h(x) =$

$\lim_{x \to -\infty} (h) = 2$

Example

Find the horizontal and vertical asymptotes of f(x). Use limits to describe the corresponding behavior.

a) $\frac{2x^2 - 1}{x^2+3}$

$\boxed{\begin{array}{l} D = \mathbb{R} \\ \text{No vert} \end{array}}$ $y=2$

$$\begin{array}{r} 2 \\ x^2+3 \overline{) 2x^2-1} \\ \underline{2x^2+6} \\ -7 \end{array}$$

$\boxed{2 + \frac{-7}{x^2+3}}$

Hor. $\lim_{x \to \infty} f(x) = 2 = \lim_{x \to -\infty} f(x)$

b) $\frac{2x + 1}{x^2 - x}$

$x = 0$

$x = 1$

$y = 0$

Domain

$x \neq 0$

$x \neq 1$

$\frac{2x}{x^2-x} + \frac{1}{x^2-x}$

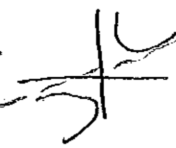
$\frac{2}{x-1} + \frac{1}{x^2-x}$

Hor. $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0$

Vert $\lim_{x \to 0^-} f(x) = \infty$ $\lim_{x \to 0^+} f(x) = -\infty$

$\lim_{x \to 1^-} f(x) = -\infty$ $\lim_{x \to 1^+} f(x) = \infty$

$\frac{\cancel{x}^3-4x}{4x}$



Slant line
AKA Oblique Asymptotes

Example

Find the asymptotes and intercepts of the function f(x) = (x³) / (x² - 4) and graph the function.

$y = x^3 + \frac{4x}{x^2-4}$

$(x-2)(x+2)$

zeros 2 & -2

- Vert Asympt. $x=2$ $x=-2$
- Slant Asymptote $y=x$
- y-int (0,0) & x int.

(Day 2) Analyzing Graphs

Example

Find the intercepts, asymptotes, use limits to describe the behavior at the vertical asymptotes, and analyze and draw the graph of the rational function

$f(x) = \frac{x+1}{x^2-x-6}$

Can something have hor + slant?

$\frac{x+1}{(x-3)(x+2)}$

$x \neq 3$ $x \neq -2$

NO Vert Asym

$x=3$

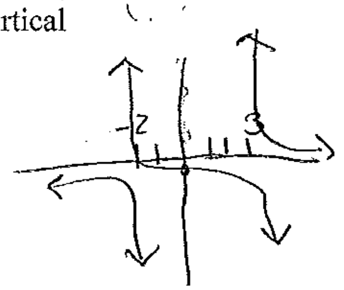
$x=-2$

Hor

$y=0$

(-1, 0)

Intercept (0, -1/6)



Limits

$\lim_{x\to 3^-} f(x) = -\infty$

$\lim_{x\to 3^+} f(x) = \infty$

$\lim_{x\to -2^-} f(x) = -\infty$

$\lim_{x\to -2^+} f(x) = \infty$

$\lim_{x\to -\infty} f(x) =$

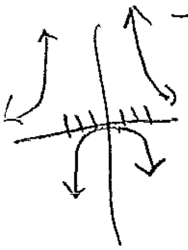
$\lim_{x\to \infty} f(x) = 0$

Example

Find the intercepts, analyze, and draw the graph of the rational function $f(x) = \frac{x^2 - 1}{x^2 - 9}$

$\frac{(x+1)(x-1)}{(x+3)(x-3)}$

Intercepts (0, 1/9)



Vert Asym $x=3$ $x=-3$

$\lim_{x\to -3^-} f(x) = \infty$ $\lim_{x\to -3^+} f(x) = -\infty$ $x \neq 3$ $x \neq -3$

$\lim_{x\to 3^-} f(x) = -\infty$ $\lim_{x\to 3^+} f(x) = \infty$

Hor $y=1$ $\lim_{x\to -\infty} f(x) = \lim_{x\to \infty} f(x) = 1$

Example

Find the end behavior asymptote of the given rational function f and graph it together with f in two windows:

a) one showing the details around the vertical asymptote(s) of f. [-10, 20] [-10, 50]
b) one showing a graph of f that resembles its end behavior asymptote. [-10, 50] [-50, 300]

$F(x) = \frac{x^2 - 2x + 3}{x - 5} = x+3+\frac{18}{x-5}$

End behavior asymptote

$y = x+3$

$$
\begin{array}{r}
x+3 \\
x-5 \overline{) x^2-2x+3} \\
x^2-5x \downarrow \\
\hline
3x+3 \\
3x-15 \\
\hline
18
\end{array}
$$

Vert

$x=5$

Example

Find the intercepts, analyze, and graph the rational function.

$$F(x) = \frac{3x^2 - 2x + 4}{x^2 - 4x + 5}$$

- $\underline{\text{Int.}}$ $(0, 4/5)$

$$\begin{array}{r} 3 \\ x^2-4x-5\overline{)3x^2-2x+4} \\ -\ 3x^2-12x-15 \\ \hline 10x+19 \end{array}$$

$$3 + \frac{10x+19}{x^2-4x+5}$$

~~[illegible]~~

- Asymptote $y=3$
- Domain $\mathbb{R}$
- Bounded
- End behavior $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 3$
- Not Symmetric
- Max $(2.44, 14.23)$
- Min $(-.25, .77)$

Try it: Find the intercepts, analyze, and graph the rational function.

$$F(x) = \frac{x^3 - 3x^2 + 3x + 1}{x - 1}$$

$$\begin{array}{r} x^2-2x+1 \\ x-1\overline{)x^3-3x^2+3x+1} \\ -\ x^3-x^2 \\ \hline -2x^2+3x \\ -2x^2+2x \\ \hline x+1 \\ x-1 \\ \hline 2 \end{array}$$

- Intercept $(0,-1)$
- Asymptote $x=1$ end behav. asymptote $y = x^2-2x+1$
- End behavior $\lim_{x\to-\infty} f(x) = \infty$ $\lim_{x\to\infty} f(x) = \infty$
- Domain $x \neq 1$
- Range $\mathbb{R}$
- ~~[illegible]~~ Unbounded

$x = 4x^2 - 24x + 36$

$4x^2 - 25x + 36 = 0$

↑ Extraneous

$x^2 = 4$

$x = \pm 2$ -2 Extraneous

2.7 Solving Equations in One Variable

- **Extraneous Solutions –** solutions of equation that were not solutions of original eqn. (so after mult or divide)

Example – Clearing Fractions

Solve 2x – 1/x = 1

LCD=x

$$x\left(\frac{2x}{1} - \frac{1}{x} = 1\right)$$

$$2x^2 - \frac{x}{x} = x$$

$$2x^2 - 1 - x = 0$$

$(2x+1)(x-1) = 0$ or use quad

$2x+1=0$ $x-1=0$

$\boxed{x = -1/2}$ $\boxed{x = 1}$

Confirm: $2(-1/2) - 1/{-1/2}$

$-1+2=1$ ✓

$2(1) - 1/1$

$2-1=1$ ✓

Solve 2x – 1/ (x-3) = 0

$$x-3\left(2x - \frac{1}{x-3} = 0\right)$$

$$2x(x-3) - \frac{x-3}{x-3} = 0$$

$$2x^2 - 6x - 1$$

$$2x^2 - 6x - 1 = 0$$

$x = -.158$ or $x = 3.158$

$2(-.158) - 1/(-.158-3) = 0$ ✓

$2(3.158) - \frac{1}{3.158-3} = 0$ ✓

Example – Extraneous Solutions

Solve the equation $\frac{x}{x-3} - \frac{4}{x-1} = \frac{8}{x^2 - 4x + 3}$

$$(x-3)(x-1)\left(\frac{x}{x-3} - \frac{4}{x-1} = \frac{8}{(x-3)(x-1)}\right)$$

$$(x^2 - x) - (4x - 12) = 8$$

$$x^2 - 5x + 12 = 8$$

$$x^2 - 5x + 4 = 0$$

$$(x-4)(x-1) = 0$$

$\boxed{x=4}$ or $x=1$

↑

Confirm:

$\frac{12}{3}$ $\frac{4}{1} - \frac{4}{3} = \frac{8}{3}$ ✓

$\frac{1}{-2} - \frac{4}{0}$ not def

$\boxed{x=1 \text{ extraneous}}$

Example – Real-World App

How much pure acid must be added to 60 mL of a 35% acid solution to produce a
Pure acid mixture that is 80% acid?

$$.35 \times 60 + 1 \times x = .80(x+60)$$

$$21 + 1x = .8x + 48$$

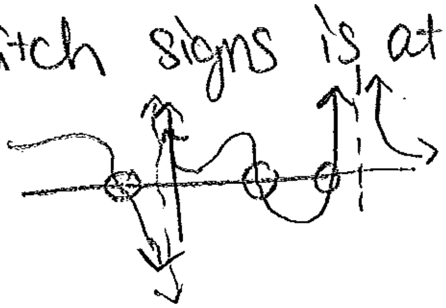
$$\underline{.2x = 27}$$

$$\boxed{x = 135\text{mL}}$$

2.8 Solving Inequalities in One Variable

- **Extraneous Solutions –**

Only way to switch signs is at zero or asymptote



Example – Finding where pos, neg, zero

Determine the x-values that cause the polynomial function to be a) zero, b) positive, and c) negative. $F(x) = (x + 7)(x + 4)(x - 6)^2$

Zeros -7, -4, 6

Determine the x-values that cause the polynomial function to be a) zero, b) positive, and c) negative. $F(x) = (x + 2)(x + 1)(x - 5)$

Zeros -2, -1, 5

Example – Solving Graphically

Solve $x^3 \le 3x + 1$

Solve $x^2 \le x + 1$

Solve $x^3 \ge 3x - 2$

$$\frac{(2x+3)}{x(x-3)}$$

Example – Sign Chart

Let r (x) = (2x + 3) / x(x – 3). Determine the values of x that cause r(x) to be (a) zero, (b) undefined. Then make a sign chart to determine the values of x that cause r(x) to be (c) positive, (d) negative.

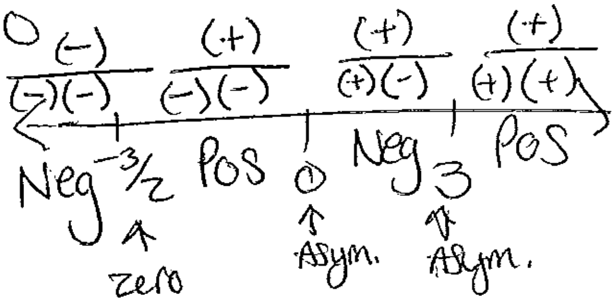
a) Zero $-3/2$

b) Undefined 0, 3

c) Pos $-3/2 < x < 0$ or $x > 3$

d) Neg $x < -3/2$ or $0 < x < 3$

$2x+3=0$



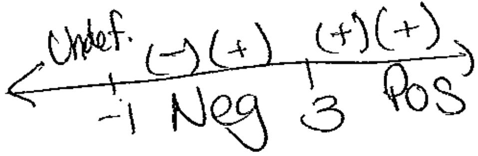
Examples- Radical and Abs. Value

Solve (x – 3) √x + 1 ≥ 0.

Undef if $x < -1$

Zeros of f 3, –1

$\{-1\} \cup [3, \infty)$

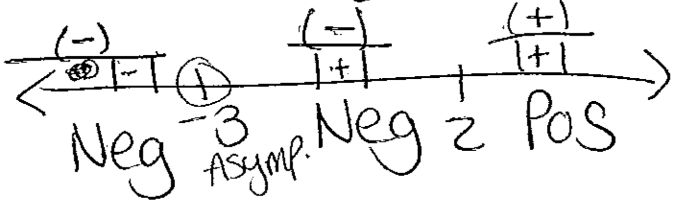


Solve (x – 2) / | x + 3 | ≤ 0.

Undef. if. $|x+3|$ $x=-3$ b/c fract.

Zero $x=2$

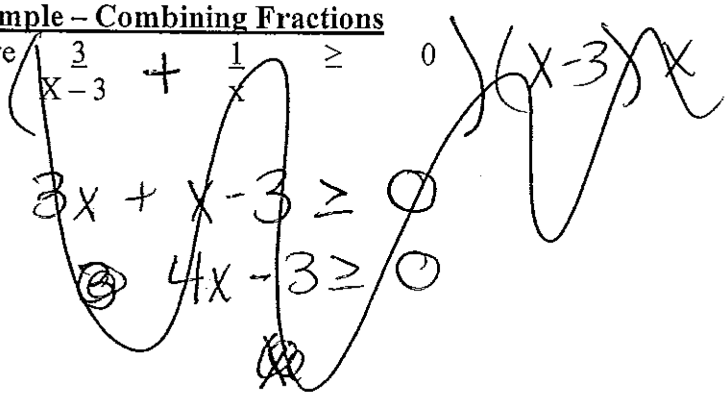
$(-\infty, -3) \cup (-3, 2]$



Look @ Graph

Example – Combining Fractions

Solve $\left(\frac{3}{x-3} + \frac{1}{x} \geq 0\right)(x-3)x$



$3x + x - 3 \geq 0$

$4x - 3 \geq 0$

$\left(\frac{3}{x-3} - \frac{1}{x} > 0\right)x(x-3)$

$\frac{3x}{x(x-3)} - \frac{(x-3)}{x(x-3)} > 0$

$\frac{3x - x + 3}{x(x-3)} > 0$

$\frac{2x+3}{x(x-3)} > 0$

$(-3/2, 0) \cup (3, \infty)$