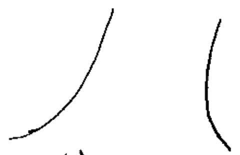
Chapter 3 Transcendental Functions

Exponential



"J" Growth "L" Decay

$y = \underline{a \cdot b}^x$

- radioactive
- Population

march madness

Round	# of Team
1	64
2	32
3	16
⋮	⋮

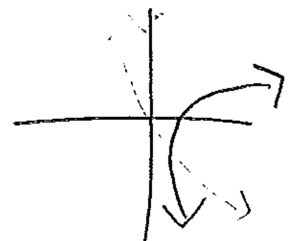
Logistic

$\frac{C}{1 + a \cdot b^x}$

- Population
- chem reactions
- Rumors
- Diseases

Logarithmic

$\ln x$



Reflect over $y=x$

- Richter Scale
- PH Scale
- Decibels

3.1 Exponential and Logistic Functions

- Exponential Function

$a \cdot b^x$ ← $b \neq 1$ Growth factor / Decay factor

↑ initial value $a \neq 0$

  - Exponential Growth $b > 1$
  - Exponential Decay $0 < b < 1$

Examples

Determine the formulas for the exponential functions whose values are given below.

a)

x	−2	−1	0	1	2
$g(x)$	$\frac{2}{25}$	$\frac{2}{5}$	2	10	50

$y = 2 \cdot 5^x$ Growth

b)

x	−2	−1	0	1	2
$g(x)$	$1\frac{3}{4}$	$3\frac{1}{2}$	7	14	28

×2 ×2

$y = 7 \cdot 2^x$ Growth

c)

x	−2	−1	0	1	2
$h(x)$	48	12	3	$\frac{3}{4}$	$\frac{3}{16}$

$y = 3 \cdot \frac{1}{4}^x$ Decay

d)

x	−2	−1	0	1	2
$h(x)$	100	20	4	0.8	0.16

$y = 4 \cdot \frac{1}{5}^x$ Decay

- The Natural Base e – $f(x) = e^x$ $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Euler introduced "oiler" $\boxed{a \cdot b^x \text{ to } a \cdot e^{kx}}$

interest compounded

- Transforming Exponential Functions $y = 2^x$
  - Right $y = 2^{x-a}$ 2^{x-3}
  - Left $y = 2^{x+3}$
  - Up $y = 2^x + 3$
  - Down $y = 2^x - 3$
  - Vertical Stretch/ Shrink — Stretch $y = 5 \cdot 2^x$ Shrink $y = \frac{1}{5} \cdot 2^x$
  - Horizontal Stretch/Shrink — stretch $y = 2^{\frac{1}{5}x}$ shrink $y = 2^{5x}$
  - Reflect y-axis $y = 2^{-x}$ x-axis $y = -2^x$

Example

Describe how to transform the graph of $f(x) = 3^x$ into the graph of the given function. Sketch the graphs by hand and support your answer with a grapher.

(a) $g(x) = 3^{x-2}$ (b) $h(x) = 3^{-x}$ (c) $k(x) = 2 \cdot 3^x$

Rt 2 Reflect over y-axis Vert Stretch 2

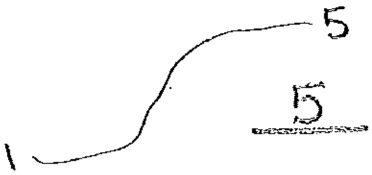
Describe how to transform $f(x) = e^x$ into the graph of the given function. Sketch the graphs by hand and support your answer with a grapher.

a) $g(x) = e^{2x}$ b) $h(x) = e^{-x}$ c) $k(x) = 3e^x$

Hor Shrink 2 Reflect over y-axis Vert. strech 3

➢ Logistic Functions → bounded restricted growth

$$f(x) = \frac{c}{1 + a \cdot b^x} \quad \text{or} \quad \frac{c}{1 + a \cdot e^{-kx}}$$



c = limit to growth

b > 1 decay ← most are growth

Example

Graph the function. Find the y-intercept and the horizontal asymptotes.

(a) $f(x) = \dfrac{2}{1 + 2 \cdot 0.5^x}$

Asym $y = 2$
$y = 0$

Int $\dfrac{2}{1 + 2 \cdot .5^0}$

$2/3$ $(0, 2/3)$

(b) $g(x) = \dfrac{15}{1 + 3 \cdot e^{-2x}}$

$y = 15$
$y = 0$

Int

$$\frac{15}{1 + 3e^{-2 \cdot 0}}$$

$$\frac{15}{1 + 3 \cdot 1} \quad 15/4$$

$(0, 15/4)$

Example - Population

Based on recent data, a logistic model for the population of Dallas, t years after 1900 is as follows:

$$P(t) = \frac{1{,}301{,}642}{1 + 21.602e^{-0.05054t}}$$

According to the model, when was the population 1 million?

[0, 120]

[-500,000, 1,500,000]

≈ 84.5

≈ 1985

3.2 Exponential and Logistic Modeling

- **Exponential Population Model** $P(t) = P_0(1+r)^t$ $P_0(1.035)^t$

  Ex Grow 3.5%
  Decay 10%
  $P_0(.9x)$

- **Radioactive Decay** – when atoms change from radioactive to non radioactive

- **Half-Life** amt of time required for ½ of radio active substance to decay

Example

Tell whether the function is exponential growth or decay and find the constant percentage rate of growth or decay.

#1-14

a) $P(t) = 3.5 \cdot 1.09^t$ Growth 9%

b) $f(x) = 78963 \cdot 0.9680^x$ Decay 3.2%

c) $h(x) = 4783 \cdot 1.32^x$ Growth 32%

Example

Suppose a culture of 200 bacteria is put into a petri dish and the culture doubles every hour. Predict when the number of bacteria will be 250,000.

#30

$y = 200 \cdot 2^x$ 10.28 hrs

$y = 250,000$

Example

Suppose the half-life of a certain radioactive substance is 24 days and there are 10 grams present initially. Find the time when there will be 2 grams of the substance remaining.

#15-18

$y = 10 \cdot \frac{1}{2}^{t/24}$

t = time in days
so half-lives is t/24

$y = 2$

55.72 days

$\boxed{\approx 56 \text{ days}}$

Example

Jonesdale High School has 630 students. Josh, Mia, Tim, and Briana start a rumor. Which spreads logistically so that $s(t) = 630/(1 + 29 \cdot e^{-0.8t})$ models the number of students who have heard the rumor by the end of t days where t=0 is the day the rumor begins to spread.

$\frac{630}{1+29e^{-.8t}}$

a) How many students have heard the rumor by the end of Day 0?

$\frac{630}{1+29} = \frac{630}{30} = \boxed{21}$

b) How long does it take for 500 students to hear the rumor?

#45-46

$y = 500$

5.89

$\boxed{5.89 \text{ days}}$

#23-28

Ex 23) Find logistic function

Initial Value = 10 limit = 40 through (1, 20)

$\frac{c}{1+a \cdot b^x}$ $c = 40$

Find a

$\frac{40}{1+a \cdot b^0}$

$\frac{40}{1+a} = 10$

$10 + 10a = 40$

$10a = 30$ $a = 3$

Find b

$\frac{40}{1+3 \cdot b^1} = 20$

$20 + 60b = 40$

$60b = 20$

$b = \frac{2}{6} = \frac{1}{3}$

$\boxed{\frac{40}{1+3 \cdot \frac{1}{3}^x}}$

Extra #25 Initial 16 max 128 through (5,32) $\dfrac{128}{1+7(.84)^x}$

$\dfrac{c}{1+a\cdot b^x}$

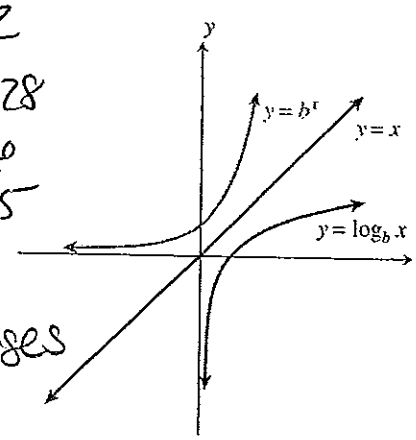
Find a
$\dfrac{128}{1+a}=16$
$16+16a=128$
$16a=112 \quad a=7$

Find b
$\dfrac{128}{1+7b^5}=32$
$32+224b^5=128$
$224b^5=96$
$b^5=3/7$
$b=.84$

3.3 Logarithmic Functions and Their Graphs

➢ **Logarithmic and Exponential Form**

$y=\log_b(x)$ iff $b^y=x$ → Inverses



Ex $\log_2 8=3 \qquad 2^3=8$

Example

Evaluate each expression.

(a) $\log_3 27$ 3 $3^? = 27$
(b) $\log_2\sqrt{2}$ 1/2
(c) $\log_4\frac{1}{16}$ -2
(d) $\log_5 1$ 0
(e) $\log_{10} 10$ 1

$y=\log x$ iff $10^y=x$

$y=\ln x$ iff $e^y=x$

Basic Properties of Logarithms

For $0<b\neq 1$, $x>0$, and any real number y.

$\log_b 1=0$ because $b^0=1$.

$\log_b b=1$ because $b^1=b$.

$\log_b b^y=y$ because $b^y=b^y$.

$b^{\log_b x}=x$ because $\log_b x=\log_b x$. ← $\log_b x = \log_b x$

Properties of Common Logarithms

Let x and y be real numbers with $x>0$.

$\log 1=0$ because $10^0=1$.

$\log 10=1$ because $10^1=10$.

$\log 10^y=y$ because $10^y=10^y$.

$10^{\log x}=x$ because $\log x=\log x$. ← $\log_{10} x = \log x$

Properties of Natural Logarithms

Let x and y be real numbers with $x > 0$.

$\ln 1 = 0$ because $e^0 = 1$.

$\ln e = 1$ because $e^1 = e$.

$\ln e^y = y$ because $e^y = e^y$.

$e^{\ln x} = x$ because $\ln x = \ln x$.

Example

Evaluate the expressions.

(a) $\log 1{,}000$ 3 10^x

(b) $\log \sqrt[3]{10}$ 1/3 10^x

(c) $\log \frac{1}{100}$ −2 10^x

(d) $10^{\log 5}$ $\log_{10} = \log_5 = 5$

(a) $\ln \sqrt[3]{e}$ 1/3

(b) $\ln e^6$ 6

(c) $e^{\ln 5}$ 5

- Activity 2 Rows

$\ln(x)$

$\ln(x+a)$ left a
$\ln(x-a)$ right a
$\ln(x)+a$ up a
$\ln(x)-a$ down a

$a\ln(x)$ vert. stretched
$\frac{1}{a}\ln(x)$ shrink

$\ln(ax)$ hor. shrink
$\ln(\frac{1}{a}x)$ hor. stretch
$\ln(-x)$ reflect over y
$-\ln(x)$ reflect over x

(Day 2 – Graphs)

Example

Describe transformations that will transform $f(x) = \ln x$ to $g(x) = 2 - 3\ln x$.

$-3\ln x + 2$

reflect over x, vert. stretch of 3, up 2

Example

Describe how to transform the graph of $y = \ln x$ or $y = \log x$ into the graph of the given function.

(a) $g(x) = \ln(x-3)$ 3 rt.
(b) $h(x) = \ln(2-x)$ ← $\ln(-x+2)$ $\ln(-(x-2))$ ← reflect over y rt 2
(c) $g(x) = 2\log x$ Vert stretch 2
(d) $h(x) = 2 + \log x$ up 2

- **Decibels -**

The level of sound intensity in **decibels** (dB) is

$$\beta = 10\log\left(\frac{I}{I_0}\right),$$

where β (beta) is the number of decibels, I is the sound intensity in W/m^2, and $I_0 = 10^{-12}$ W/m^2 is the threshold of human hearing (the quietest audible sound intensity).

Ex Table 280
Sound intensity of subway

$B = 10\log\frac{10^{-2}}{10^{-12}}$

$10\log 10^{10}$

$10 \cdot 10$

$\boxed{100}$

We've done $\log_{10} x = 2$.

Now: ~~loge300~~ $\log_4 7$

3.4 Properties of Logarithmic Functions

$b^x \cdot b^y$

Let b, R, and S be positve real numbers with $b \neq 1$, and c any real number.

Product rule: $\log_b(RS) = \log_b R + \log_b S$ $\quad a^x \cdot a^y = a^{x+y}$

Quotient rule: $\log_b\left(\frac{R}{S}\right) = \log_b R - \log_b S$ $\quad a^x / a^y = a^{x-y}$

Power rule: $\log_b(R)^c = c\log_b R$ $\quad (a^x)^y = a^{xy}$

Ex

$\log(2 \cdot 4) = \log 2 + \log 4$

$\log\left(\frac{8}{2}\right) = \log 8 - \log 2$

$\log 2^3 = 3\log 2$

Example

#1-12

Assuming x and y are positive, use properties of logarithms to write $\log(16x^2y^3)$ as a sum of logarithms or multiples of logarithms.

$\log 16 + \log x^2 + \log y^3$

$\log \underset{2^4}{16} + 2\log x + 3\log y$

$\boxed{4\log 2 + 2\log x + 3\log y}$

Example

Assuming x is positive, use properties of logarithms to write $\ln \frac{\sqrt{x^2-3}}{x}$ as a sum or difference of logarithms or multiples of logarithms.

$\ln\sqrt{x^2-3} - \ln x$

$\boxed{\frac{1}{2}\ln(x^2-3) - \ln x}$

#51 Still using $\beta = 10 \log \frac{I}{I_0}$

#13-22 Addit. **Example**

Assuming x and y are positive, use properties of logarithms to write $\ln x^4 - 3\ln(xy)$ as a single logarithm.

Ex

$\ln x^2 - \ln(xy)$

$\ln x^2 - \ln xy$

$\ln \frac{x^2}{xy} = \boxed{\ln \frac{x}{y}}$

$\ln x^4 - \ln(xy)^3$

$\ln x^4 - \ln x^3y^3$

$\ln \frac{x^4}{x^3y^3}$

$\boxed{\ln \frac{x}{y^3}}$

#23-28 Ex

➢ **Change of Base Formula** – purpose is to change log so has diff base.

$\log_4 7$ $\quad \frac{\ln 7}{\ln 4}$

$4^x = 7$

$\ln 4^x = \ln 7$

$x \ln 4 = \ln 7$

$x = \frac{\ln 7}{\ln 4}$

$\log_b x = \frac{\log_a x}{\log_a b}$

Example

#39-46 Describe how to transform the graph of $f(x) = \ln x$ into the graph of the given function. Sketch the graph by hand and support your answer with a grapher.

(a) $g(x) = \log_3 x$ (b) $h(x) = \log_{1/3} x$

$\frac{\ln x}{\ln 3} = \ln x \cdot \frac{1}{\ln 3}$

$= \ln x \cdot .91$

Vert Shrink

$\frac{\ln x}{\ln 1/3} = \frac{\ln x}{\ln 1 - \ln 3} = \frac{\ln x}{-\ln 3} = -\frac{1}{\ln 3}\ln x$

0

reflect over x & vert shrink .91

Ex Solve

$\log_6 10$

$6^x = 10$

$\ln 6^x = \ln 10$

$x \ln 6 = \ln 10$

$x = \frac{\ln 10}{\ln 6}$

$\boxed{1.28}$

➢ **Re-expressing Data** – transforming data

#54 **Example**

The relationship between intensity I of light (in lumens) at a depth of x feet in Lake Erie is given by $\log \frac{I}{12} = -.00235x$. What is the intensity at a depth of 40 feet?

9.16 lumens

$\log \frac{I}{12} = -.00235 \cdot 40$

$\log \frac{I}{12} = -.094$

$\log I - \log 12 = -.094$

Recall
Solve $3^x = 6$

$\log_3 6$ $\ln(3^x)$ $\ln 6$

$x \ln 3 = \ln 6$

$x = \ln 6 / \ln 3$

Now: $20(1/2)^{x/3} = 5$

3.5 Equation Solving and Modeling

Sometimes logarithmic equations can be solved by changing to exponential form.

For any exponential function $f(x) = b^x$,

If $b^u = b^v$, then $u = v$.

For any logarithmic function $f(x) = \log_b x$,

If $\log_b u = \log_b v$, then $u = v$.

#13

Example

Solve $27(2/3)^{x/4} = 12$.

$\frac{2}{3}^{x/4} = \frac{4}{9}$

$\frac{2}{3}^{x/4} = \frac{2}{3}^2$

$\frac{x}{4} = 2$

$\boxed{x = 8}$

Solve $\log x^4 = 4$.

1 to 1 prop $\cancel{\log x^4 = \log 10^4}$

$x^4 = 10^4$

$x^4 = 10{,}000$

~~10000000000~~

10 or -10

Quad Exp

$10^4 = x^4$

$10{,}000 = x^4$

$x = 10$ or -10

Check w/ Table or Graph

Graph Int $y=5$ $x = 2.31$

Solve $(e^x - e^{-x})/2 = 5$

$e^x \cdot$ $e^x - e^{-x} = 10 \cdot e^x$ ← so get quad

$(e^x)^2 - 10e^x - e^0$

$(e^x)^2 - 10e^x - 1 = 0$ $m^2 - 10m - 1$ $5 \pm \sqrt{26}$ No neg

Note: When you solve logarithmic equations be to keep track of the domain!

* Be careful
So Sol. are all there

$e^x = 5 + \sqrt{26}$

$\ln e^x = \ln(5 + \sqrt{26})$

$x = 2.312$

Example

Solve $\ln(x-3) + \ln(x+4) = 3\ln 2$

$\ln(x-3) + \ln(x+4) - 3\ln 2 = 0$

$\boxed{\text{Graph} \quad x = 4}$

$\ln((x-3)(x+4)) = \ln 2^3$

$x^2 + x - 12 = 8$

$x^2 + x - 20 = 0$

$(x+5)(x-4)$ $x = 4$

#1-17 odd on white boards

$2\log x = 6$

$\log x = 3$

$10^3 = x$

$x = 1000$

$\log x^2 = 6$

$x^2 = 10^6$

$x = \pm 10^3$ No neg can't be neg

$\boxed{x = \pm 1000}$

#50 Day 2

Newton's Law $T(t) = T_m + (t_0 - T_m)e^{-kt}$

T_m: temp surrounding med.

t_0: Initial temp

Example – Real World

A hard-boiled egg at temperature 97°C is placed in 17°C water to cool. Three minutes later the temperature of the egg is 48°C. Use Newton's Law of Cooling to determine when the egg will be 20°C.

$48 = 17 + \underbrace{80}_{(97-17)} e^{-3K}$

$31 = 80e^{-3K}$

$\ln \frac{31}{80} = \cancel{\ln e}^{-3K}$

$\ln \frac{31}{80} = -3K$

$K = \frac{\ln 31/80}{-3}$

$K = .316$

$20 = 17 + 80e^{-.316t}$

$3 = 80e^{-.316t}$

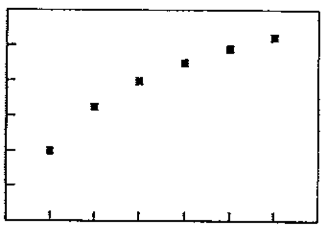
$\ln 3/80 = -.316t$

$t = 10.39$

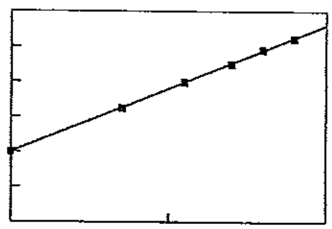
10.39 min

➢ Logarithmic Re-expression

1. Natural Logarithmic Regression Re-expressed: $(x, y) \to (\ln x, y)$



[0, 7] by [0, 30]
(x, y) data
(a)

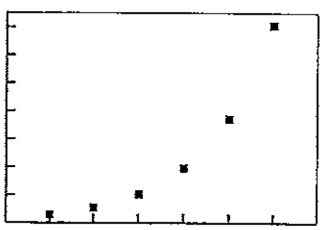


[0, 2] by [0, 30]
$(\ln x, y) = (u, y)$ data with linear regression model
$y = au + b$
(b)

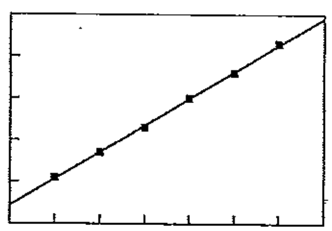
Conclusion:

$y = a \ln x + b$ is the logarithmic regression model for the (x, y) data.

2. Exponential Regression Re-expressed: $(x, y) \to (x, \ln y)$



[0, 7] by [0, 75]
(x, y) data
(a)

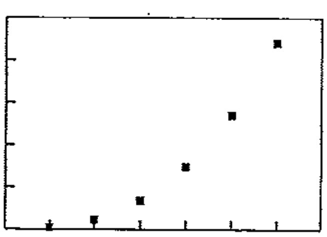


[0, 7] by [0, 5]
$(x, \ln y) = (x, v)$ data with linear regression model
$v = ax + b$
(b)

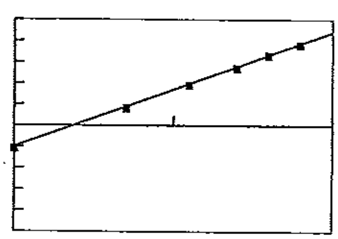
Conclusion:

$y = c(d^x)$, where $c = e^b$ and $d = e^a$, is the exponential regression model for the (x, y) data.

3. Power Regression Re-expressed: $(x, y) \to (\ln x, \ln y)$



[0, 7] by [0, 50]
(x, y) data
(a)



[0, 2] by [−5, 5]
$(\ln x, \ln y) = (u, v)$ data with linear regression model
$v = au + b$
(b)

Conclusion:

$y = c(x^a)$, where $c = e^b$, is the power regression model for the (x, y) data.

Ex #45

Formula

$\log \frac{a_1}{a_2} = R_1 - R_2$

$\log \frac{a_1}{a_2} = 1.3$

≈ 20X Greater

Ex #47

$pH = -\log[H^+]$

a) water 1.26×10^{-4}

amonia ≈ 11 1.26×10^{-11}

b) $\frac{1.26 \times 10^{-4}}{1.26 \times 10^{-12}}$

10^{8}

c) 8

- Semiannually - 2x
- Annually - 1x
- Quarterly - 4x
- Monthly 12x

3.6 Mathematics of Finance

➢ **Compounded Interest –** interest earns interest

Annual

$A = P(1+r)^n$

$A = P\left(1+\frac{r}{k}\right)^{kt}$ rate per comp. period; # of compounding periods

Example

Paul invests $500 at 7% compounded annually. Find the value of his investment 10 years later.

$500(1.07)^{10} = \boxed{\$983.58}$

Robert invests $500 at 9% annual interest compounded monthly. Find the value of his investment 5 years later.

$500\left(1+\frac{.09}{12}\right)^{12(5)} \boxed{\$782.84}$

Mazie has $600 to invest at 8% annual interest compounded monthly. How long will it take for her investment to grow to $2400?

$2400 = 600\left(1+\frac{.08}{12}\right)^{12t}$

$y = 2400$

$4 = 1.00\bar{6}^{12t}$

$\ln 4 = 12t \ln 1.00\bar{6}$

$t = 17.38$ yrs

Steven has $500 to invest. What annual interest rate compounded quarterly is required to double his money is 10 years?

$1000 = 500\left(1+\frac{r}{4}\right)^{4(10)}$

$2 = \left(1+\frac{r}{4}\right)^{40}$

$\ln 2 = 40 \ln\left(1+\frac{r}{4}\right)$

$.0173 = \ln\left(1+\frac{r}{4}\right)$

Graph

$[0, .15]$ $[-500, 1500]$

.069

6.9%

➢ **Compounded Continuously –**

$A = P\left(1+\frac{r}{k}\right)^{kt}$ or $A = Pe^{rt}$

k times a yr. continuously

Example

Suppose Moesha invests $200 at 7% annual interest compounded continuously. Find the value of her investment at the end of each of the years 1, 2, . . . , 7.

$200e^{.07t}$

1	214.50
2	230.05
3	246.74
4	264.63
5	283.81
6	304.39
7	326.46

- **Annual Percentage Yield (APY)** – the percentage rate, when compounded annually the yields the same return

Example

Janet invests $2000 with Crab Key Bank at 5.15% annual interest compounded quarterly. What is the equivalent APY?

$$2000\left(1+\frac{.0515}{4}\right)^4 = 2000(1+x)$$

$$1.0525 = 1+x \quad x = .0525$$

$$5.25\%$$

- **Annuity** – equal periodic payments
- **Ordinary** – deposits at end of period @ same time int. posted
- **Future Value** – $R\,\frac{(1+i)^n-1}{i}$
- **Present Value** – $R\,\frac{1-(1+i)^{-n}}{i}$

i = int. rate per period
R = dollars
n = # of payments

Example

At the end of each quarter year, Emile makes a $400 payment into the Smithville Financial Fund. If his investments earn 7.75% annual interest compounded quarterly, what will be the value of Emile's annuity in 25 years?

$$FV = R\,\frac{(1+i)^n - 1}{i}$$

$$= 400\,\frac{(1+.0775/4)^{100} - 1}{\frac{.0775}{4}}$$

$25 \cdot 4$

Exact $120,029.57

$\boxed{\$120{,}030}$

Example

Carlos purchases a pickup truck for $18,500. What are the monthly payments for a 4-year loan with a $2000 down payment if the annual interest rate is 2.9%?

$$16500 = R\,\frac{1-(1+.029/12)^{-48}}{.029/12}$$

$$\cancel{18{,}500}\;16500 = R\left(\frac{1-(1+.029/12)^{-4\cdot 12}}{.029/12}\right)$$

18500 − 2000

$$16500\left(\frac{.029}{12}\right) = R\,(1-(1+.029/12)^{-48})$$

$$39.875 = R\,(1-(1+.029/12))^{-48}$$

$\boxed{\$364.49}$

18500−2000

$$16500 = R\,\frac{1-(1+.029/12)^{-48}}{.029/12}$$

$$16500(.029/12) = R\;1-(1+.029/12)^{-48}$$

$$39.875 = R \cdot .109$$

$$\boxed{R = 364.49}$$