6.1 Vectors in the Plane

- **Two dimensional Vector** – ordered pair of $\mathbb{R}$ in component form $\langle a, b \rangle$
  - Standard Rep. – from origin to (a, b)
  - magnitude = length of arrow
  - Direction = where pting to
  - zero vector = zero length & no direction
- **Head Minus Tail Rule**

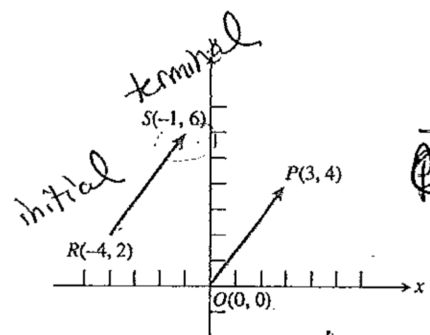
(x_1, y_1) initial $\quad (x_2, y_2)$ terminal

Vector $\langle x_2 - x_1, y_2 - y_1 \rangle$

Example

Show the arrow from R to S is equivalent to the arrow from O to P.

#2-4



$\overrightarrow{RS}\ \langle -1-(-4), 6-2 \rangle = \boxed{\langle 3, 4 \rangle}$

$\overrightarrow{OP}\ \langle 3-0, 4-0 \rangle = \boxed{\langle 3, 4 \rangle}$

- **Magnitude** – dist. formula

$|v| = \sqrt{a^2 + b^2}$

Example

#5-12

Find the magnitude of v represented by $\overrightarrow{PQ}$, where $P = (3, -4)$ and $Q = (5, 2)$.

$\sqrt{(5-3)^2 + (2-(-4))^2}$ OR $\langle 2, 6 \rangle$ vector

$2^2 + 6^2 \qquad \sqrt{2^2 + 6^2}$

$\sqrt{40} \quad \sqrt{4 \cdot 10} \quad 2\sqrt{10} \qquad \sqrt{40}$ mag

- **Vector Addition and Scalar Multiplication** – Let $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$ be vectors and let k b[e] a real number (scalar). The **sum (or resultant) of th[e] vectors u and v** is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle.$$

The **product of the scalar** k and the vector **u** is

$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$

Example

#13-20

Let $\mathbf{u} = \langle 3, 2 \rangle$ and $\mathbf{v} = \langle 5, -1 \rangle$. Find the component form of the following vectors:

(a) $2\mathbf{v}$ (b) $\mathbf{u}+\mathbf{v}$ (c) $(-2)\mathbf{u}+3\mathbf{v}$

$\boxed{\langle 10, -2 \rangle}$

$\boxed{\langle 8, 1 \rangle}$

$\langle -6, -4 \rangle + \langle 15, -3 \rangle$

$\boxed{\langle 9, -7 \rangle}$

$\frac{3}{|5|} = \frac{1}{|5|} \cdot 3$

(Day 2)

- **Unit Vector** – vector w/ length 1 which means $u = \frac{v}{|v|} = \frac{1}{|v|} v$

mag.

Example

#21-24

Find a unit vector in the direction of $\mathbf{v} = \langle 5, -1 \rangle$.

mag $\sqrt{5^2 + (-1)^2} = \sqrt{26}$

$\frac{v}{|v|}$ $\langle 5, -1 \rangle \cdot \frac{1}{|\sqrt{26}|} = \left\langle \frac{5}{\sqrt{26}}, \frac{-1}{\sqrt{26}} \right\rangle$

$\left\langle \frac{5}{\sqrt{26}} i + \frac{-1}{\sqrt{26}} j \right\rangle$

- **Direction Angle** – θ, made w/ pos x-axis

- **Resolving the Vector** –

"solving"

$V = \langle |v|\cos\theta, |v|\sin\theta \rangle$

x y

Example

#29-32

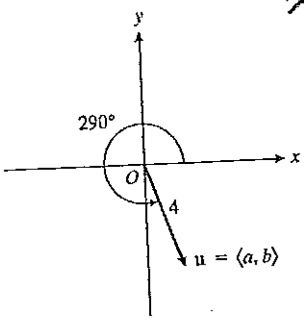


Figure 6.1 The direction angle of u is 290°.

Find the components of a vector **u** with direction angle 290° and magnitude 4 (Figure 6.1).

$\langle 4 \cdot \cos 290, 4 \sin 290 \rangle$

$\langle 1.37, -3.7 \rangle$

Example

#33-38

Find the magnitude and direction angle of each vector:

(a) $\mathbf{u} = \langle 1, 4 \rangle$

mag $\sqrt{1^2+4^2}$ = $\sqrt{17}$

$\frac{1}{\sqrt{17}} = \frac{\sqrt{17}\cos\alpha}{\sqrt{17}}$

$\cos^{-1}\frac{1}{\sqrt{17}} = \cos^{-1}\cos\alpha$

Dir. $\alpha = 75.96°$

(b) $\mathbf{v} = \langle -5, 4 \rangle$

mag $\sqrt{41}$

$\frac{-5}{\sqrt{41}} = \frac{\sqrt{41}\cos\beta}{\sqrt{41}}$

$= 141.34°$

(c) $\mathbf{w} = \langle 3, -2 \rangle$

mag $\sqrt{13}$

$\cos^{-1}\frac{3}{\sqrt{13}} = \cos^{-1}\frac{\sqrt{13}\cos C}{\sqrt{13}}$

$C = 33.69°$

$360 - 33.69 = 326.31°$

C, β, α, ⟨1,4⟩, ⟨3,-2⟩

Example – Application

An airplane is flying on a compass heading (bearing) of 340 degrees at 325 mph. A wind is blowing with the bearing 320 degrees at 40 mph.

mag.

Bearing deg. from due North

a) Find the component form of the velocity of the plane.
b) Find the actual ground speed and direction of the plane.

angle from x-axis

70° 340

$340 - 180 - 90 = 70°$

a) $\langle 325\cos 110, 325\sin 110 \rangle$

$= \langle -111.16, 305.4 \rangle$ plane

130° 50°

b) wind. direction 130°

mag. 40 $\langle 40\cos 130, 40\sin 130 \rangle$

$= \langle -25.71, 30.64 \rangle$ wind

336.04 θ -136.86

Plane + wind $= \langle -136.86, 336.04 \rangle$

$\tan^{-1}\left(\frac{336.04}{-136.86}\right)$

$\theta = 67.84$

+270 = 340°

317.85 mph

6.2 Dot Product of Vectors

- **Dot Product/Inner Product** – (scalar) cos of angle between 2 vectors * length of each vector

$u = \langle u_1, u_2 \rangle \quad v = \langle v_1, v_2 \rangle$

$u \cdot v = u_1 v_1 + u_2 v_2$

- **Properties of the Dot Product**

① $u \cdot v = v \cdot u$ (Commutative)

② $u \cdot u = |u|^2$

③ $0 \cdot u = 0$

④ $u(v+w) = uv + uw$ Dist

⑤ $c \cdot uv = (cu) \cdot v = (cv)u$

Associative

Example

#1-8

Find the dot product of the given vectors.

(a) $\mathbf{u} = \langle 3, 1 \rangle$, $\mathbf{v} = \langle 2, -5 \rangle$ $\quad 3 \cdot 2 + 1 \cdot -5 = 6 + -5 = 1$

(b) $\mathbf{u} = \langle 5, 4 \rangle$, $\mathbf{v} = \langle -4, 5 \rangle$ $= 0$

(c) $\mathbf{u} = (-2\mathbf{i} + 2\mathbf{j})$, $\mathbf{v} = (3\mathbf{i} - 5\mathbf{j})$ $= -2 \cdot 3 + 2 \cdot -5$

$-6 + -10 = -16$

Example

#9-12

Use the dot product to find the length of the vector u= <4, -3>

Rev. $\sqrt{4^2 + -3^2}$

$\sqrt{16+9}$

$\sqrt{25}$

$= 5$

$|\langle 4, -3 \rangle|$ dot prod.

$\sqrt{\langle 4,-3 \rangle \cdot \langle 4, -3 \rangle}$

$\sqrt{4 \cdot 4 + -3 \cdot -3}$

$\sqrt{16+9}$

$\sqrt{25} = 5$

- **THEOREM: Angle Between Vectors**

If θ is the angle between the nonzero vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

and $$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

Recall Yesterday

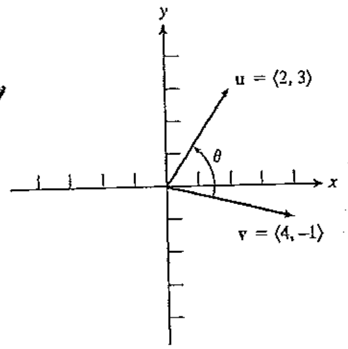
$v = \langle |v| \cos\theta, |v| \sin\theta \rangle$

$\cos\theta \cdot |u| \cdot |v| = (u\ v)$

$|u| = \sqrt{2^2+3^2}$
$= \sqrt{13}$ $\quad \sqrt{17}$

Example

#18-22



Find the angle between the vectors $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle 4, -1\rangle$ (Figure 6.3).

$\cos\theta = \dfrac{u \cdot v}{|u||v|} = \dfrac{8 + -3}{\sqrt{13}\ \sqrt{17}} = \dfrac{5}{\sqrt{13}\sqrt{17}}$

$\cos\theta = \dfrac{5}{\sqrt{13}\sqrt{17}} = \boxed{70.3°}$

➤ **Orthogonal Vectors –** Perp.
Dot Product of Zero b/c Cos 90 = 0

> Parallel Vectors are "mult." of each other
$\langle 2, 5\rangle$
$\langle 4, 10\rangle$

#23-24 **Example**

Prove that the vectors u = <2, 3> and v = <-6, 4> are orthogonal.

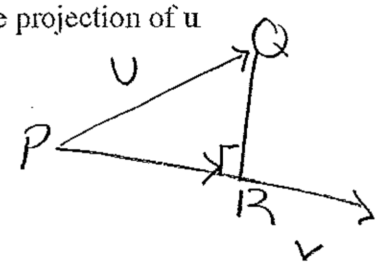
$u \cdot v$
$2 \cdot -6 + 3 \cdot 4$
$\cancel{-6} +$
$-12 + 12 = 0$ ✓

Day 2 – Projection and Work

➤ **Projection of u onto v -** If **u** and **v** are nonzero vectors, the projection of **u** onto **v** is

$$\text{proj}_v u = \left(\frac{u \cdot v}{|v|^2}\right) v.$$

dropping perp. line from one to another



#25-28 **Example**

(a) Find the projection of vector $\mathbf{u} = \langle 4, 3\rangle$ onto vector $\mathbf{v} = \langle 6, 2\rangle$.

(b) Write **u** as a sum of two orthogonal vectors, one of which is $\text{proj}_v\mathbf{u}$.

a) $\dfrac{\langle 4,3\rangle \cdot \langle 6,2\rangle}{\sqrt{40}^2}$

$= \dfrac{24+6}{40} = \dfrac{30}{40}$ ✓

$= \frac{3}{4}\langle 6, 2\rangle = \boxed{\left\langle \frac{9}{2}, \frac{3}{2}\right\rangle}$

b) $U = U_1 + U_2$
$\langle 4,3\rangle = \left\langle \frac{9}{2}, \frac{3}{2}\right\rangle + U_2$
$U_2 = \left\langle -\frac{1}{2}, \frac{3}{2}\right\rangle$

So $U = \left\langle -\frac{1}{2}, \frac{3}{2}\right\rangle + \left\langle \frac{9}{2}, \frac{3}{2}\right\rangle$

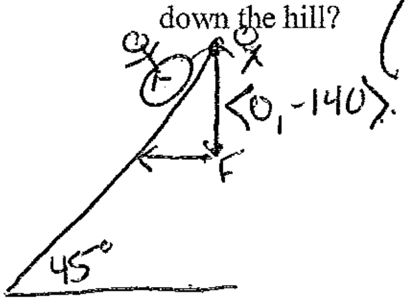
Example

#46

incl.

Juan is sitting on a sled on the side of a hill ~~included~~ at 45 degrees. The combined weight of Juan and the sled is 140 pounds. What force is required for Rafaela to keep the sled from sliding down the hill?

(we calc this force downhill, force is opp)



$$\text{Proj}_v F = \frac{F \cdot v}{|v|^2} v$$

$$\frac{F \cdot v}{1^2} v$$

$V = \cos 45, \sin 45$

unit circle $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$|V| = 1$ mag

$$F \cdot v$$

$$0 \cdot \frac{\sqrt{2}}{2} + -140 \frac{\sqrt{2}}{2}$$

$$\frac{-140\sqrt{2}}{2} = -98.9$$

99 lbs

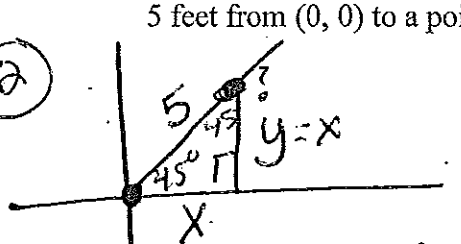
- **Work -** If F is a constant force whose direction is the same as the direction of $\overrightarrow{AB}$, then the **work** W done by F in moving an object from A to B is $W = |F||\overrightarrow{AB}|$

#50, 52, 53

Example

Find the work done by a force F of 50 pounds acting in the direction <2, 3> in moving an object 5 feet from (0, 0) to a point in the first quadrant along the line y= x.

(2)



(1) $50 \dfrac{\langle 2,3 \rangle}{|\langle 2,3 \rangle|} = \dfrac{50}{\sqrt{13}} \langle 2,3 \rangle$

$$x^2 + y^2 = 5^2$$

$$\frac{\cancel{2}x^2}{\cancel{2}} = \frac{25}{2}$$

$$x^2 = 12.5$$

$$x = \sqrt{12.5}$$

(3) $W = F \cdot \overrightarrow{AB}$

$$\frac{50}{\sqrt{13}} \langle 2,3 \rangle \cdot \langle \sqrt{12.5}, \sqrt{12.5} \rangle$$

$$\left\langle \frac{100}{\sqrt{13}}, \frac{150}{\sqrt{13}} \right\rangle \cdot \langle \sqrt{12.5}, \sqrt{12.5} \rangle$$

pounds

6.3 Parametric Equations and Motion

- **Recall: Parametric Equations**

when x & y are both defined in terms of 3rd variable

$x = \cos t$
$y = \sin t$

- **Parametric Curves**

The graph of the ordered pairs (x,y) where

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are functions defined on an interval I of t-values is a parametric curve. The equations are parametric equations for the curve, the variable t is a parameter, and I is the parameter interval.

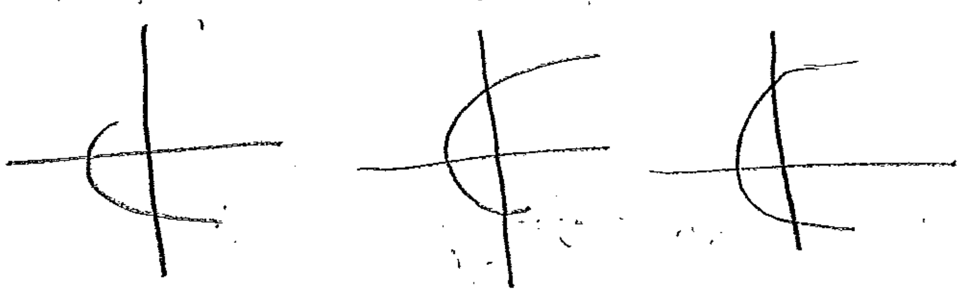
Example

#1-10

For the given parameter interval graph the parametric equations.

$$x = t^2 - 2, \; y = 3t$$

a) $-3 \le t \le 1$ b) $-2 \le t \le 3$ c) $-3 \le t \le 3$



Example

#11-26

Eliminate the parameter and identify the graph of the parametric curve

$x = 3t - 2, \quad y = 1 - 2t, \quad -\infty < t < \infty.$

Solve for t

$X = 3t - 2$

$X + 2 = 3t$

$\frac{1}{3}(X + 2) = t$

$1 - 2\left(\frac{1}{3}(x+2)\right)$

$1 - 2\left(\frac{1}{3}x + \frac{2}{3}\right)$

$1 - \frac{2}{3}x - \frac{4}{3}$

$y = -\frac{2}{3}x - \frac{1}{3}$

Line

Example

Eliminate the parameter and identify the graph of the parametric curve

$$x = t^2 - 2, \quad y = 3t$$

$t = y/3$

$$x = \left(\frac{y}{3}\right)^2 - 2$$

$$x = \frac{y^2}{9} - 2$$

$$x + 2 = y^2/9$$

$$9(x+2) = y^2$$

$$y = \sqrt{9(x+2)}$$

$$y = -\sqrt{9(x+2)}$$

side ways parabola

Example

Eliminate the parameter and identify the graph of the parametric curve

$x = 3\cos t, \quad y = 3\sin t; \quad 0 \le t \le 2\pi.$

$$x^2 + y^2 = 9\cos^2 t + 9\sin^2 t$$

$$= 9(\cos^2 t + \sin^2 t)$$

$$1$$

$$x^2 + y^2 = 9$$

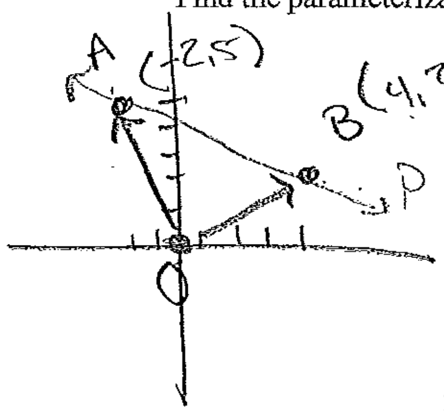
circle radius 3

$$y = \sqrt{-x^2+9}$$

$$y = -\sqrt{-x^2+9}$$

#27-32 **Example**

Find the parameterization for the line through the points (-2, 5) and (4, 2).



$$OP = OA + AP$$

$$+ t \cdot AB$$

$$\langle x, y \rangle \quad \langle -2, 5 \rangle + t\langle 4 - -2, 2 - 5 \rangle$$

$$\langle -2, 5 \rangle + t\langle 6, -3 \rangle$$

$$\langle x, y \rangle = \langle -2, 5 \rangle + \langle 6t, -3t \rangle$$

$$\boxed{\begin{aligned} x &= -2 + 6t \\ y &= 5 - 3t \end{aligned}}$$

OR

$$OP = AP + OB$$

$$t\langle -2-4, 5-2 \rangle + \langle 4, 2 \rangle$$

$$t\langle -6, 3 \rangle + \langle 4, 2 \rangle$$

$$x = -6t + 4$$

(Day 2) – Motion

$x_1 = 20t$ $y_1 = 3$

$x_2 = 24t - 10$ $y_2 = 5$

Example – pg. 483 #37

a) [0, 300]
b) [0, 10]
Ben ahead
2 ft
62
60

Example

A toy rocket is launched straight up from a cart. The toy's initial position is 3 ft above ground and its initial velocity is 25 ft/sec. Graph the toy's height against time, give the height of the toy above the ground, and simulate the toy's motion for each length of time.

(a) 0.5 sec (b) 0.75 sec (c) 1 sec (d) 1.25 sec 9.25

11.5 12.75

Motion
$16t^2 + V_0 t + S_0$

$-16t^2 + V_0 t + S_0$

$y = -16t^2 + 25t + 3$

$x = t$

V_0 = initial vel
S_0 = starting

Example #9

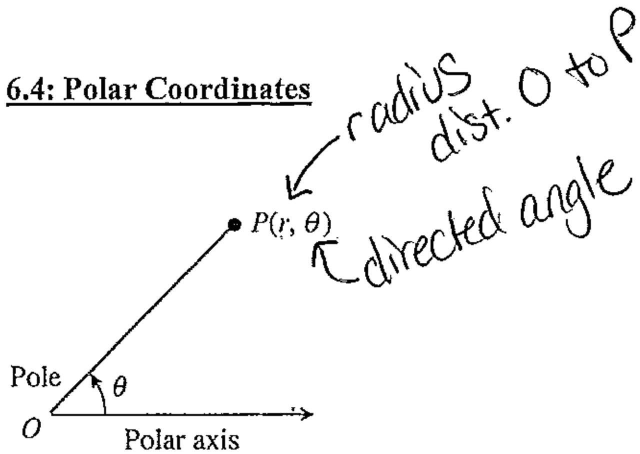
Kevin hits a baseball at 3 ft above the ground with an initial speed of 150 ft/sec at an angle of 18 degrees with the horizontal. Will the ball clear a 20-ft wall that is 400 ft away?

$x_1 = (150 \cos 18)t$

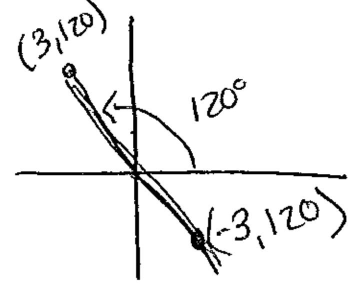
$y_1 = -16t^2 + (150 \sin 18)t + 3$

No

6.4: Polar Coordinates



$(3, 120°)$



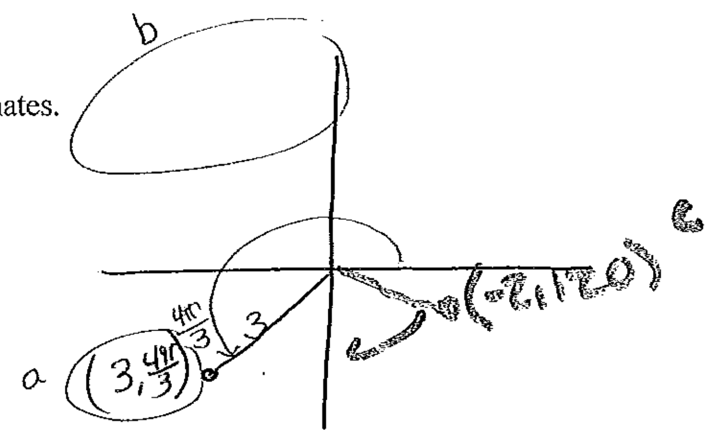
$(-3, 120°)$

Example

#7-14

Plot the points with the given polar coordinates.

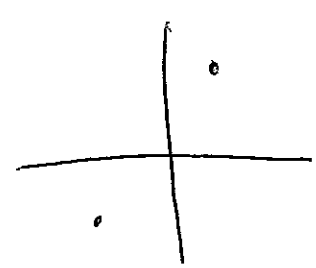
a) $(3, 4\pi/3)$ 240°
b) $(-3, 17\pi/10)$ 306°
c) $(-2, 120°)$



Let the point P have polar coordinates (r, θ). Any other polar coordinate of P must be of the form

$$(r, \theta + 2\pi n) \text{ or } (-r, \theta + (2n+1)\pi)$$

where n is any integer. In particular, the pole has polar coordinates $(0, \theta)$, where θ is any angle.



Example

#23-26

Find all polar coordinates for point P if point P has polar coordinates $(3, \pi/3)$.

$(3, \pi/3 + 2\pi) = (3, 7\pi/3)$ $(3, \pi/3 + 2\pi n)$

1 rot

$(-3, \pi/3 + \pi n)$

odd #

It is important to be able to go from rectangular to polar coordinates.

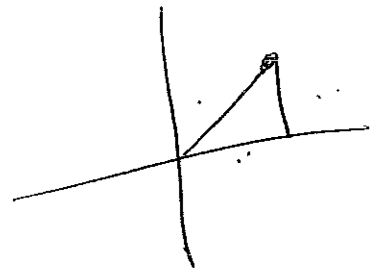
Let the point P have polar coordinates (r, θ) and rectangular coordinates (x, y). Then

Pol to Rect

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}.$$

Rect to Pol.



#1-4
#15-22

Example

Find the rectangular coordinates of the points with the given polar coordinates.

(a) $P(4, 2\pi/3)$ 120° (b) $Q(5, -75°)$

r θ

$x = 4\cos 120°$
$= -2$

$y = 4\sin 120°$
~~3.46~~
$\boxed{(-2, 3.46)}$

$x = 5\cos -75$
1.29

$y = 5\sin -75$
~~-4.83~~
$\boxed{(1.29, -4.83)}$

#27-30

Example

Find two polar coordinate pairs for the points with the given rectangular coordinates.

(a) $P(1, 2)$ (b) $Q(-4, 4)$

x y

1. $x^2 + y^2 = r^2$
$1^2 + 2^2 = r^2$
$5 = r^2$
$r = \sqrt{5}$

2. $\tan\theta = \frac{y}{x}$
$\tan\theta = \frac{2}{1}$
63.43° 1.1 rad

$(\sqrt{5}, 63.43°)$
or
$(\sqrt{5}, 1.1 \text{ rad})$

1. $-4^2 + 4^2 = r^2$
$r = \sqrt{32}$
$4\sqrt{2}$

2. $\tan\theta = \frac{4}{-4}$
$\tan\theta = -1$
$-.785$ rad.
$-45°$

$(\sqrt{32}, -.785)$
$(\sqrt{32}, -45°)$

Stop

#36-42

Example

Convert $r = 4\cos\theta$ to rectangular form and identify the graph. Support your answer with a polar graphing utility.

Goal to r^2, $r^2 = x^2 + y^2$

$(r = 4\cos\theta)r$ mult by r

$r^2 = 4r\cos\theta$

Sub $x^2 + y^2 = 4x$

complete the square $\left(\frac{b}{2}\right)^2$ $\left(\frac{-4}{2}\right)^2$ 4

$x^2 - 4x + 4 + y^2 = 0 + 4$

$\boxed{(x-2)^2 + y^2 = 4}$

Circle Center $(2, 0)$ Radius 2

Example

Convert $(x-2)^2 + (y+2)^2 = 8$ to polar form. Then graph the polar equation. $r=$

#43-50

Expand $x^2 - 4x + 4 + y^2 + 4y + 4 = 8$

$x^2 + y^2 - 4x + 4y = 0$

$r^2 - 4x + 4y = 0$

Sub $r^2 - 4r\cos\theta + 4r\sin\theta = 0$

$r(r - 4\cos\theta + 4\sin\theta) = 0$

$r = 0$ or $r - 4\cos\theta + 4\sin\theta = 0$

$r - 4\cos\theta + 4\sin\theta = 0$

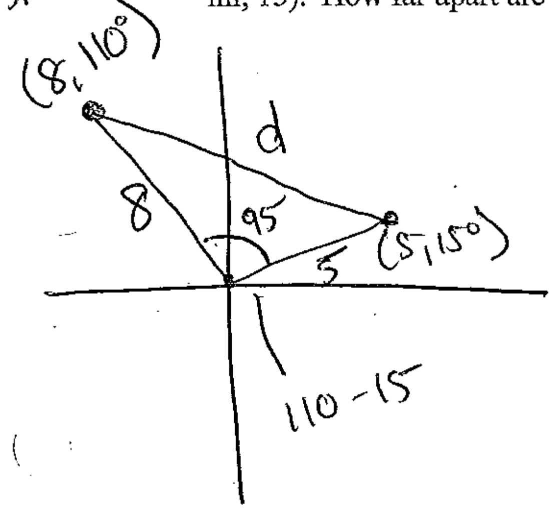
$\boxed{r = 4\cos\theta - 4\sin\theta}$

Center $(2, -2)$ Radius $\sqrt{8}$

Example

#52

Radar detects two airplanes at the same altitude. Their polar coordinates are (8 mi, 110°) and (5 mi, 15°). How far apart are the airplanes?



$d^2 = a^2 + b^2 - 2ab\cos\theta$

$d^2 = 8^2 + 5^2 - 2 \cdot 8 \cdot 5\cos 95$

$d = 9.8$ mi

6.5: Graphs of Polar Equations

- **Symmetry Tests**
  - **X-axis** (Polar Axis) A
  - **Y-axis** (line $\theta = \frac{\pi}{2}$) B
  - **Origin** (Pole) C

$r = a \pm b\sin\theta$

① $\frac{a}{b} < 1$ Loop

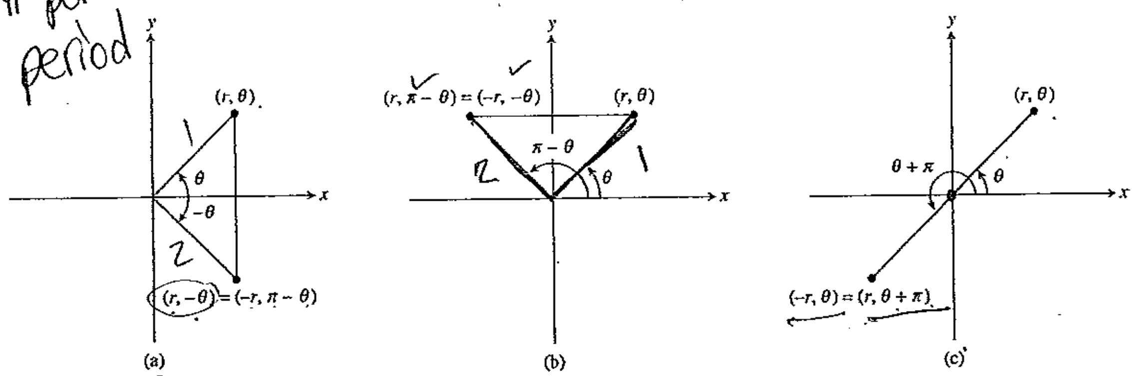
② $\frac{a}{b} = 1$ Cardioid

③ $1 < \frac{a}{b} < 2$ Dimpled

④ $\frac{a}{b} \geq 2$ Convex

sin (y-axis)
cos (x-axis)

$y = a\sin(n\theta)$
n even 2π period
n odd π period



Stop
#14-20

Example

Use the polar symmetry tests to determine if the graph of $r = -3\cos 2\theta$ is symmetric about the x-axis, y-axis, or the origin.

x-axis $(r, -\theta)$

$r = -3\cos 2(-\theta)$

$= -3\cos(-2\theta)$ cos even

$= -3\cos 2\theta$

∴ symm. x-axis
same r started w/

y-axis $(-r, -\theta)$

$-r = -3\cos 2(-\theta)$

$= -3\cos(-2\theta)$

$r = -3\cos 2\theta$

$r \neq -r$

y-axis $(r, \pi - \theta)$

$r = -3\cos 2(\pi - \theta)$

$= -3\cos 2\pi - 2\theta$

$= -3\cos -2\theta$

$= -3\cos 2\theta$

Yes

Origin $(r, \theta + \pi)$

$r = -3\cos 2(\theta + \pi)$

$= -3\cos 2\theta + 2\pi$ Period 2π

$= -3\cos 2\theta$

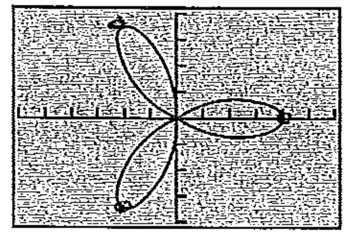
Example

#21-24

"Identify the points on the graph of $r = 4\cos 3\theta$ for $0 \le \theta \le 2\pi$ (Figure 6.15) that give maximum r-values.

$y = |4\cos 3x|$

$0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$



[−6, 6] by [−4, 4]

Figure 6.15 The graph of $r = 4\cos 3\theta$ appears to have three maximum points, but there are actually six of them for $0 \le \theta \le 2\pi$ because the grapher traces the curve twice.

➤ **Rose Curve –** Polar eqns. $r = -3\sin 2\theta$

$r = a\cos n\theta$ $r = a\sin n\theta$ n integer >1

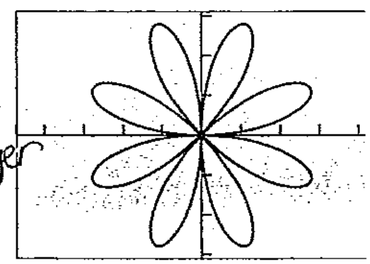
n is odd, same petals

n is even, 2·n petals

Rng. $[-|a|, |a|]$

smallest K value

n odd	π
n even	2π



[−4.7, 4.7] by [−3.1, 3.1]

➤ **Limacon Curves -** Polar eqns.

"Snail"

$r = a \pm b\sin\theta$

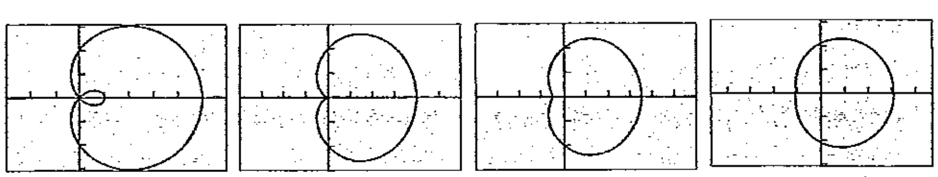
$a+b > 0$

y-axis

$r = a \pm b\cos\theta$

x-axis

Rng: $[a-b, a+b]$



Limaçon with an inner loop: $\frac{a}{b} < 1$ Cardioid: $\frac{a}{b} = 1$ Dimpled limaçon: $1 < \frac{a}{b} < 2$ Convex limaçon: $\frac{a}{b} \ge 2$

#26-34

Example

Analyze the graph of the rose curve $r = 3\cos 2\theta$.

Symmetry: x, y, origin
max r-value: 3
Domain: $(-\infty, \infty)$
Rng: $[-3, 3]$
Continuous
Bounded
No Asymptotes

2.2
4 petals

Example

Analyze the graph of $r = 3 - 3\sin\Theta$

Symm: y-axis
Max r: 6
Domain: $(-\infty, \infty)$
Rng: $[0, 6]$
3-3 3+3

Cont.
Bounded
No Asymptotes

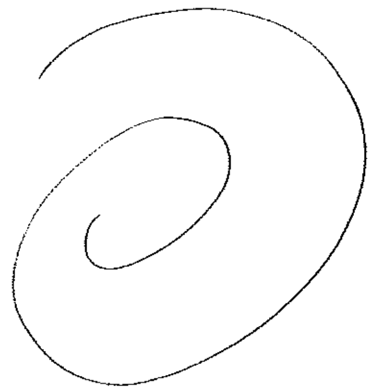
Example

Analyze the graph of the limaçon curve $r = 1 + 2\sin\theta$.

Symmetry: y-axis
Max r: 3
Domain: $(-\infty, \infty)$
Rng: $[-1, 3]$
1-2 1+2
Cont.
Bounded
No Asymptotes

Example

Analyze the graph of r = Θ. ← spiral of Archimedes



Domain: $(-\infty, \infty)$

Rng: $(-\infty, \infty)$

Continuous

Unbounded

No max r-value

No asymptotes

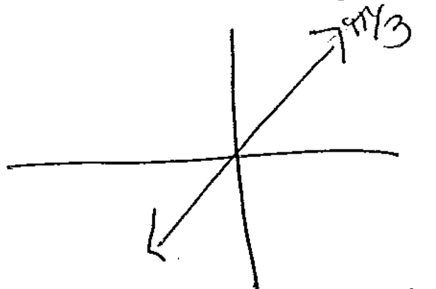
Symm. y-axis

window

$[-1000, 1000]$

Ex $\Theta = 4\pi/3$

$= 60°$



Domain: $4\pi/3 + 2\pi n$

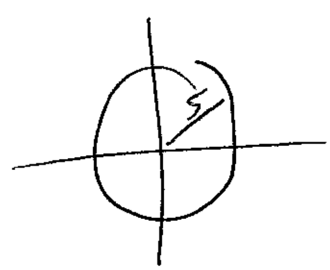
Rng $(-\infty, \infty)$

Symm. Origin

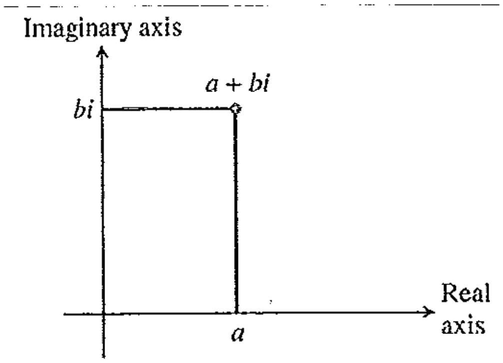
Unbounded

Max r none

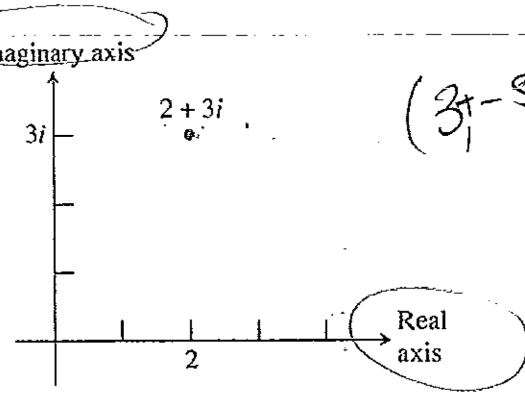
Ex $r = 5$



6.6 De Moivre's Theorem and nth Roots



(a)



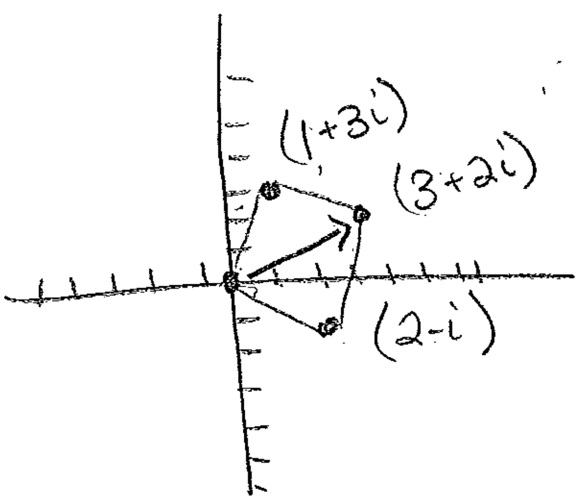
(b)

$(3i - 5i)$

Example

#1-2

Plot $u = 1+3i$, $v = 2 - i$, and $u + v$ in the complex plane. The three points and the origin determine a quadrilateral. Is it a parallelogram?



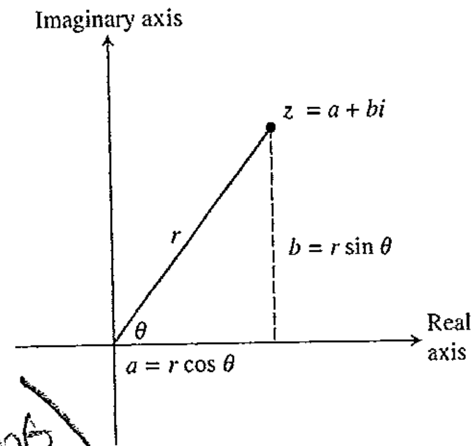
$U + V = 1 + 3i + 2 - i$
$= 3 + 2i$

Yes

Vector Addition!

➤ **Modulus -** The **absolute value** or **modulus** of a complex number $z = a + bi$ is $|z| = |a+bi| = \sqrt{a^2 + b^2}$.
In the complex plane, $|a+bi|$ is the distance of $a+bi$ from the origin.

➤ **Trigonometric Form -**



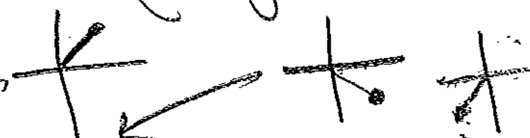
Ex

$r(\cos\theta + i\sin\theta)$

8 / 20°

$8(\cos 20 + i\sin 20)$ think of vectors

(mag · cos θ, mag · sin θ)



c) $r = \sqrt{29}$

$\tan\theta = 5/2$

$\theta = 1.19 + \pi$ 3rd quad

4.33 $\sqrt{29}(\cos 4.33 + i\sin 4.33)$

#3-10 **Example**

Find the trigonometric form with $0 \le \theta < 2\pi$ for the complex number.

Stand. (a) $2\sqrt{3} + 2i$ (b) $5 - 5i$ (c) $-2 - 5i$

a) $r = \sqrt{(2\sqrt{3})^2 + 2^2}$

$\sqrt{16} = 4$

$\tan\theta = \frac{2}{2\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \pi/6$

$\boxed{4(\cos \pi/6 + i\sin \pi/6)}$

b) $r = \sqrt{5^2 + (-5)^2}$

$= \sqrt{50} = 5\sqrt{2}$

$\tan\theta = \frac{-5}{5} = -1$

$\theta = \pi/4$

$\theta = -\pi/4 + 2\pi$

$= \frac{7\pi}{4}$

$\boxed{5\sqrt{2}\left(\cos \frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)}$

#19-22 **Example**

Express the product of z_1 and z_2 in standard form:

Similar exp. rules

$z_1 = 12\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, $z_2 = \sqrt{6}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$.

$\frac{\pi}{6} + \frac{-\pi}{4} = \frac{2\pi}{12} + \frac{-3\pi}{12} = \frac{-\pi}{12}$

$5x^2y^3 \cdot 2x^4y^5$ $z_1 \cdot z_2$

$10x^6y^8$

$12\sqrt{3} \cdot \sqrt{6}\left(\cos\left(\frac{\pi}{6} + \frac{-\pi}{4}\right) + i\sin\left(\frac{\pi}{6} + \frac{-\pi}{4}\right)\right)$

$36\sqrt{2}\left(\cos -\pi/12 + i\sin -\pi/12\right)$ Trig form

$49.18 - i13.18$

$49.18 - 13.18i$

$$\frac{10x^7y^3}{5x^4y} = 2x^3y^2$$

Example

#27-30

Express the quotient z_1/z_2 in standard form:

$$z_1 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right), \quad z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right).$$

$$\frac{4\pi}{3} - \frac{5\pi}{4}$$
$$\frac{4\pi}{12} - \frac{15\pi}{12} = -\frac{11\pi}{12}$$

$$\frac{4}{\sqrt{2}}\left(\cos\left(\frac{4\pi}{3} - \frac{5\pi}{4}\right) + i\sin\left(\frac{4\pi}{3} - \frac{5\pi}{4}\right)\right)$$

$$2\sqrt{2}\left(\cos -\frac{11\pi}{12} + i\sin -\frac{11\pi}{12}\right) \text{ Trig}$$

$$\boxed{-2.73 - .732i} \text{ Stand.}$$

De Moivre's Theorem (De mov)

Pull Out Unit Circle

$$z = r\cdot(\cos\theta + i\sin\theta)$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example

#32-38

Find $[(-\frac{\sqrt{2}}{2}) + i(\frac{\sqrt{2}}{2})]^8$ using De Moivre's Theorem

cos sin

$\frac{3\pi}{4}$ Unit Circle

$$r = \sqrt{\left(\frac{-\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$$

$$z^8 = 1^8\left(\cos 8\cdot\frac{3\pi}{4} + i\sin 8\cdot\frac{3\pi}{4}\right)$$

$$= 1\left(\cos 6\pi + i\sin 6\pi\right)$$

$$= 1\cdot(1 + i0)$$

$$\boxed{1 + 0i}$$

Roots

$$\sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n}+i\sin\frac{\theta+2\pi k}{n}\right)$$

$k=0,1,2\ldots n-1$

#39-50 **Example**

Find the four fourth roots of $2(\cos(2\pi/3)+i\sin(2\pi/3))$

$$\sqrt[4]{2}\left(\cos\frac{\frac{2\pi}{3}+2\pi k}{4}+i\sin\frac{\frac{2\pi}{3}+2\pi k}{4}\right)$$

$k=0,1,2,3$

$\frac{2\pi}{3}\div 4$

$\frac{2\pi}{3}\cdot\frac{1}{4}=\frac{2\pi}{12}=\frac{\pi}{6}$

$$\boxed{\begin{aligned}
z_1&=\sqrt[4]{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\\
z_2&=\sqrt[4]{2}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\\
z_3&=\sqrt[4]{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)\\
z_4&=\sqrt[4]{2}\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)
\end{aligned}}$$

Plug in zero for K

$\frac{\pi}{6}+\frac{3\pi}{6}$ $\frac{4\pi}{6}$

$\frac{4\pi}{6}+\pi$

$4\pi/6$ $\frac{3\pi/2}{}$ $\frac{6\pi}{6}$ $\frac{10\pi}{6}$

Example

X Find the cube roots of -1 and plot them.

Trig form $\underset{\cos}{-1}+\underset{\sin}{0}i$ π $K=0,1,2$

$$z_1=\cos\frac{\pi+2\pi k}{3}+i\sin\frac{\pi+2\pi k}{3}$$

$$=\cos\pi/3+i\sin\pi/3=\boxed{\frac{1}{2}+i\frac{\sqrt{3}}{2}}$$

$$z_2=\cos 3\pi/3+i\sin 3\pi/3=\boxed{-1+0i}$$ Real

$$z_3=\cos\frac{5\pi}{3}+i\sin 5\pi/3=\boxed{\frac{1}{2}-i\frac{\sqrt{3}}{2}}$$

Example

Express the roots of unity in standard form a + bi. Graph each root in the complex plane.

Sixth roots of unity.