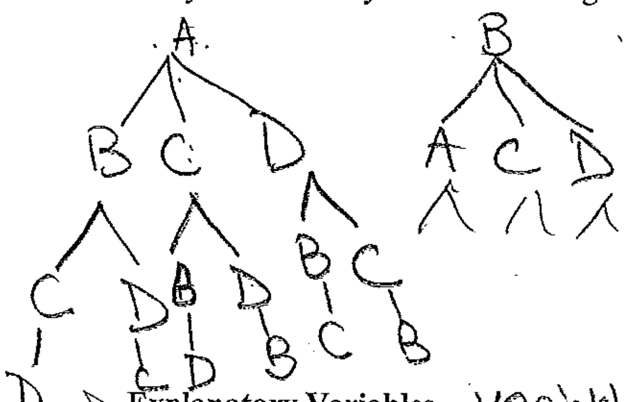
9.1 Basic Combinatorics

Example

In how many different ways can four distinguishable items be arrange in order?



$\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$ math PRB #4

$4!$ ← factorial

$\boxed{24 \text{ ways}}$

- **Explanatory Variables** – Variables that affect another (manipulated)
- **Response Variables** – Variables that is affected
- **The Multiplication Principle of Counting** –

Item 1 occurs in m ways, and items 2 occurs in n ways, then m*n # of ways to makes 1st then second selection.

Example

A New York State license plate consists of three letters of the alphabet followed by four numerical digits (0 through 9). Find the number of different license plates that can be formed

L L L # # # #

letters 26

digits 10

(a) if there is no restriction on the letters or numbers that can be used;

(b) if no letter or digit can be repeated.

a) $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^4 = \boxed{175,760,000 \text{ ways}}$

b) $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = \boxed{78,624,000 \text{ ways}}$

#math # $_nP_r$

- **Permutations** – arrangement of elements, order matters

Position

n objects $_nP_r$ ← r taken at a time

$$\frac{n!}{(n-r)!}$$

Ex

Ways to arrange lineup w/ 9 ppl?

$_9P_9$

Ex

→ 10ppl. on team

$_{10}P_9$ $10 \cdot 9 \cdot 8 \ldots$

Example

Count the number of different 7-letter "words" (don't worry about whether they're in the dictionary) that can be formed using the letters in each word.

(a) FLORIDA $7! = \boxed{5040}$

(b) ALABAMA $7!/4!$ $\frac{7\cdot6\cdot5\cdot\cancel{4!}}{\cancel{4!}} = 210$ 4 As Repeat

(c) MONTANA

$_7P_7$

$7!/(2!\,2!) = \boxed{1,260}$

b/c
2 As
2 Ns

Example

Evaluate each expression without a calculator.

(a) $_6P_2$ (b) $_{10}P_6$ (c) $_nP_5$ ← $n\cdot(n-1)(n-2)(n-3)(n-4)$

1 2 3 4

6 items
2 spots

$6\cdot5$ $10\cdot9\cdot8\cdot7\cdot6\cdot5$

30 151,200

or $\frac{6!}{4!}$ $\boxed{6\cdot5}$ $\frac{10!}{4!}$ $\frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot\cancel{4!}}{\cancel{4!}}$

Example

Sixteen actors answer a casting call to try out for roles as dwarfs in a production of Snow White and the Seven Dwarfs. In how many different ways can the director cast the seven roles?

$_{16}P_7$

1 2

$16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10$

$\boxed{57,657,600}$

(Day 2)

4.4.4

4^{20}

➤ **Combinations** – arrangement of elements, order does not matter

Ex
5 ppl. sitting A B C D E
3 chairs
$5\cdot4\cdot3 = 60$ combos

ABC
ACB
BAC

same thing

6
3!

$\frac{_nP_r}{r!}$ $\frac{\frac{n!}{(n-r)!}}{r!}$

$_nC_r$
"n choose r"

$\binom{n}{r}$

$= \frac{n!}{}$

Example

In each of the following scenarios tell whether permutations (ordered) or combinations (unordered) are being described.

a) A president, VP, and Secretary are chosen from a 25-member Foreign Language Club.

Perm, order matters

b) A cook chooses 5 potatoes from a bag of 12 potatoes to make a potato salad.

Combination, same salad

c) A teacher makes a seating chart for 22 students in a classroom with 30 desks.

Perm, diff order = new chart

d) Locker Com. = Perm, order matters

e) 5 members of Bball team Combo.

Example

In a beauty pageant, 50 contestants must be narrowed down to 15 finalists. In how many possible ways can the fifteen finalists be selected?

$$_{50}C_{15} \quad \frac{50!}{15!\,35!} \quad 50-15$$

$$2,250,829,575,000$$

Example

A coin is tossed 20 times and the heads and tails sequence is recorded. From among all the possible sequences of heads and tails, how many have exactly seven heads?

$$_{20}C_7 = \boxed{77,520}$$

Example

Professor Indiana Jones gives his class 20 study questions, from which he will select 8 to be answered on the final exam. How many ways can he select the questions?

Combin.

$$_{20}C_8 \quad \frac{20!}{8!\,12!} = 125,970$$

> Subsets
2^n = subsets of set w/ n objects

Example

A national hamburger chain used to advertise that it fixed its hamburgers "256 ways" since patrons could order whatever toppings they wanted. How many toppings must have been available?

$$2^n = 256$$

$$n \ln 2 = \ln 256$$

$$n = 8$$

9.2 The Binomial Theorem

If you expand $(a+b)^n$ for $n = 0, 1, 2, 3, 4,$ and 5, here is what you get:

$$
\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= 1a^1b^0 + 1a^0b^1\\
(a+b)^2 &= 1a^2b^0 + 2a^1b^1 + 1a^0b^2\\
(a+b)^3 &= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3\\
(a+b)^4 &= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4\\
(a+b)^5 &= 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5
\end{aligned}
$$

Example

a's down
b's up

#1-4 Expand $(a+b)^7$, using a calculator to compute the binomial coefficients.

${}_7C_x \ \{0,1,2,3,4,5,6,7\}$

$$\boxed{1a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + 1b^7}$$

- **Pascal's Triangle –**

Row Zero

1
1 1
1 2 1
1 3 3 1

Coeff. binomial powers

Example

#5-8 Show how row 5 of Pascal's triangle can be used to obtain row 6 and use the information to write the expansion of (x + y)^6

1 5 10 10 5 1
1 6 15 20 15 6 1

$1x^6 + 6x^5y + 15x^4y^2 \ldots$

- **Recursion Formula for Pascal's Triangle**

$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ or, equivalently,

${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$

$\binom{10}{3} = {}_{10}C_3$

$\frac{10!}{3!\,7!}$

Example

#13-15

Find the coefficient of x^{13} in the expansion of $(x+4)^{17}$.

${}_{17}C_{13}\, x^{13}\, 4^4 \leftarrow 17-13$

$2380 \cdot x^{13}\, 4^4$

$2380 \cdot x^{13} \cdot 256$

$609{,}280x^{13}$

coeff x^{10} in exp. $(x+2)^{15}$

${}_{15}C_{10} \cdot x^{10}\, 2^5$

$96{,}096\, x^{10}$

Stop

Example

#17-26

Expand $(4x - y^4)^3$.

a b

$_3C_x$ 1, 3, 3, 1 $a = 4x$ $b = -y^4$

$1a^3 + 3a^2b + 3ab^2 + b^3$ even

$(4x)^3 + 3(4x)^2(-y^4) + 3(4x)(-y^4)^2 + (-y^4)^3$

$3 \cdot 16x^2 \cdot -y^4$ $12x \cdot 1y^8$

$\boxed{64x^3 - 48x^2y^4 + 12xy^8 - y^{12}}$

Expand

$(2x - y^2)^4$

a b

1 4 6 4 1

$1a^4b^0 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

$= 1(2x)^4(-y^2)^0 + 4(2x)^3(-y^2) + 6(2x)^2\underbrace{(-y^2)^2}_{\text{pos}} +$

$4(2x)(-y^2)^3 + \underbrace{(-y^2)^4}_{\text{even}}$

$= 16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8$

- **Factorial Identities**

① For $n \geq 1$

$n! = n(n-1)!$

② For $n \geq 0$

$(n+1)! = (n+1) \cdot n!$

$$ {}_nC_r \quad \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

\# 29 + 34

Example

Prove that $\binom{n+1}{2} - \binom{n}{2} = n$ *for all integers* $n \geq 2$

$$= \frac{(n+1)!}{2!(n+1-2)!} - \frac{n!}{2!(n-2)!}$$

$$= \frac{(n+1)!}{2!(n-1)!} - \frac{n!}{2!(n-2)!}$$

$$= \frac{(n+1)(n)\cdot\cancel{(n-1)!}}{2!\cancel{(n-1)!}} - \frac{n\cdot(n-1)\cancel{(n-2)!}}{2!\cancel{(n-2)!}}$$

$$= \frac{n^2+n}{2!} - \frac{n^2-n}{2!} = \frac{\cancel{n^2}+n-\cancel{n^2}+n}{2}$$

$$= \frac{2n}{2} = n$$

Ex

$$\binom{n}{2} + \binom{n+1}{2} = n^2$$

$$\frac{n!}{2!(n-2)!} + \frac{(n+1)!}{2!(n-1)!}$$

$$= \frac{n\cdot(n-1)\cdot\cancel{(n-2)!}}{2!\cancel{(n-2)!}} + \frac{(n+1)n\cdot\cancel{(n-1)!}}{2!\cancel{(n-1)!}}$$

$$= \frac{n^2-n}{2!} + \frac{n^2+n}{2!}$$

$$= 2n^2$$

9.3 Probability

- **Sample Space** – set of all possible outcomes

- **Event** – subset of sample space (rolling a 3)

- **Probability** $\dfrac{\text{\# of outcomes event}}{\text{\# of outcomes sample space}}$ $\dfrac{1}{6}$

- **Probability Distribution** – collection of probabilities of outcomes in sample space

2 Dice sum $6 \cdot 6$

Outcome	Prob.
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

Example

Find the probability of each of the following events:

#1-8

a) Tossing a head on one toss of a fair coin 1/2 T H
b) Tossing two heads in a row on two tosses of a fair coin {TT, HT, TH, HH} 1/4 $\frac{1}{2} \cdot \frac{1}{2}$
c) Drawing a queen from a standard deck of 52 cards $\frac{4}{52} = \frac{1}{13}$
d) Rolling a sum of 4 on a single roll of two fair dice $3/36 = 1/12$

e) Guessing all 6 numbers in a state lottery that requires you to pick 6 numbers between 1 and 46, inclusive

$_{46}C_6$

9,366,819

$\dfrac{1}{9,366,819}$

Example

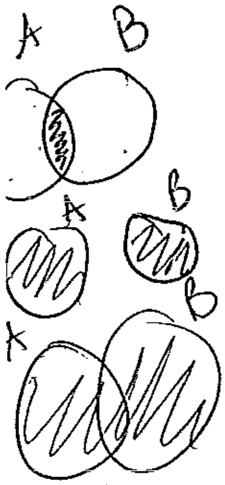
Find the probability of rolling a sum divisible by 4 on a single roll of two fair dice.

$6 \times 6 = 36$ possible

Sum div. by 4: 4, 8, 12

$3/36 + 5/36 + 1/36 \quad 9/36 = 1/4$

- **Probability Function** – function where each outcome has real # value must add to 100%
- **Probability of an Event (Outcomes are not Equally Likely)** All add to 100%



> Probability Independent Events $P(A \text{ and } B) = P(A) \cdot P(B)$

> Probability Mutually Exclusive $P(A \text{ or } B) = P(A) + P(B)$

> Probability Inclusive Event $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

P(4 or mult of 2)

don't double count middle

Example

#9

It is possible to weight a standard 6-sided die in such a way that the probability of rolling each number n is exactly $1/(n^2 + 1)$?

No

Outcome	Probability
1	1/2
2	1/5
3	1/10
4	1/17
5	1/26
6	1/37

$\frac{1}{1^2+1}$

$\frac{1}{2^2+1}$

$\frac{1}{3^2+1}$

$\cancel{\frac{1}{(-1)^2+1}}$ $\cancel{\frac{1}{2} P(-1)}$

Should add to 100%

.92 ≠ 1

#23-26

Example

Sal opens a box of two-dozen chocolate crèmes and generously offers three of them to Val. Val likes vanilla crème the best, but all the chocolates look alike on the outside. If 11 of the 24 cremes are vanilla, what is the probability that all three of Val's picks turn out to be vanilla?

Sample $_{24}C_3 = 2024$

Event Vanilla $_{11}C_3 = 165$

$\frac{165}{2024} = \frac{15}{184} = 8.2\%$

Example

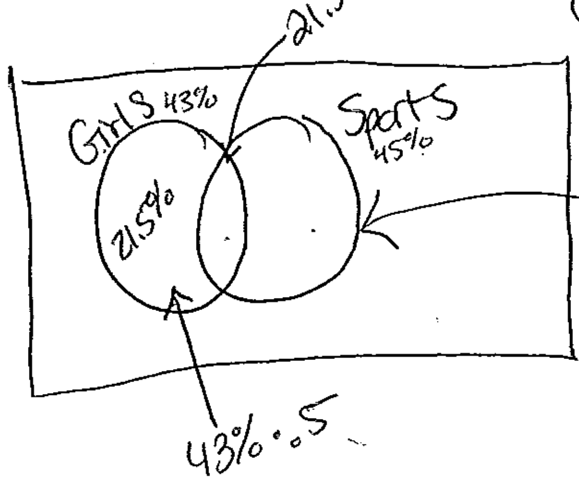
In a large high school, 43% of the students are girls and 45% of the students play sports. Half of the girls at the school play sports.

(a) What percentage of the students who play sports are boys? 23.5%

(b) If a student is chosen at random, what is the probability that it is ~~a boy who~~ does not play sports?

33.5%

#27



Sports Girls Sports

45% − 21.5% = 23.5% Boys Sports

b) (100 − 43% Girls) − 23.5%

% Boys − % Boys Sports

57%

Example

If it rains tomorrow, the probability is 0.8 that John will practice his piano lesson. If it does not rain tomorrow, there is only a 0.4 chance that John will practice. Suppose that the chance of rain tomorrow is 60%. What is the probability that John will practice his piano lesson?

P(John will Pract) = will rain No rain

$(.8)(.60) + (.40)(.40) = .64$

100%

64%

- **Conditional Probability**

$P(B|A)$

"P of A given B"

$$= \frac{P(A \text{ and } B)}{P(A)}$$

Example

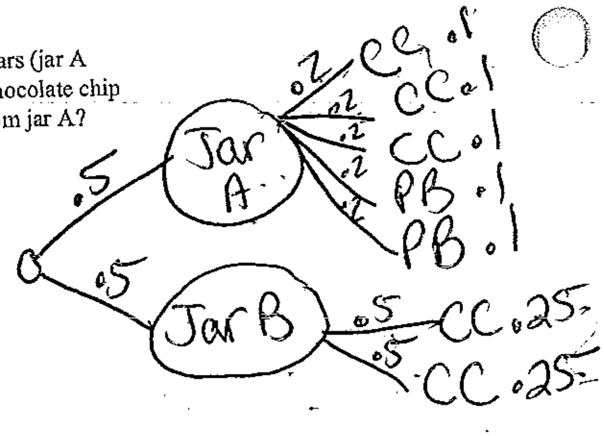
Suppose we have drawn a cookie at random from one of two identical cookie jars (jar A contains 3 chocolate chip and 2 peanut butter cookies, while jar B contains 2 chocolate chip cookies). Given that it is chocolate chip, what is the probability that it came from jar A?

P(Jar A | c chip)

CC Jar A

$$\frac{1/2 \cdot 3/5}{P\,CC} \quad \frac{.3}{.8} = \frac{3}{8}$$

CC



Example

Suppose Michael makes 90% of his free throws. If he shoots 20 free throws, and if his chance of making each one is independent of the other shots, what is the probability that he makes

a) All 20? B) Exactly 18? C) At least 18?

a) $(.90)^{20} = .12158$ 12.2%

b) $\binom{20}{18}(.9)^{18}(.10)^{2} = .285$ 28.5%

$_{20}C_{18}$

c) P(18) + P(19) + P(20)

$.285 + \binom{20}{19} \cdot (.9)^{19}(.1)^{1} + .122 = .6769 =$ 67.7%

.27

We roll a fair die four times. Find the probability that we roll:
(a) all 5's. (b) no 5's. (c) exactly three 5's.

9.4 Sequences

- **Sequence** - ordered progression of #s
  - **Finite Sequence** — where domain fixed n (stops)
  - **Infinite Sequence** — where domain not fixed n all $\mathbb{N}$

Example

Find the first 6 terms and the 100th term of the sequence $\{a_k\}$ *in which* $a_k = k^2 - 1$

#1-4

$a_6 = 6^2 - 1 = 35$ $a_{100} = 9{,}999$

$a_5 = 24$

$a_4 = 15$

$a_3 = 8$ $1^2 - 1$

$a_2 = 3$ $a_1 = 0$

#5-10 **Example**

Find the first 6 terms and the 100th term for the sequence defined recursively by the conditions:

using previous terms

$$b_1 = 3$$

$$b_n = b_{n-1} + 2 \; for \; n > 1$$

$b_2 = 3 + 2 = 5$

$b_3 = 5 + 2 = 7$

$b_4 = 9$

$b_5 = 11$

$b_6 = 13$

Turn into explicit

$b_{100} = (2 * 100) + 1$

$= 201$

- **Limit of a Sequence**
  - If $\{a_n\}$ is a seq. and $\lim_{n \to \infty} a_n = L$ (finite #), then sequence converges and has a limit
  - No limit, diverges

#11-20

Example

Determine whether the sequence converges or diverges. If it converges, give the limit.

(a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots, \frac{1}{2n}, \ldots$ (b) $\frac{11}{1}, \frac{12}{2}, \frac{13}{3}, \frac{14}{4}, \ldots$ $\frac{(n+10)}{n}$

(c) 5, 10, 15, 20, 25, … (d) 0.1, 0.2, 0.3, 0.4, 0.5, …

a) Converges to Zero
b) Converges to ~~Zero~~ One
c) Diverges
d) Diverges

Example

Determine whether the sequence converges or diverges. If it converges, give the limit.

a) $\left\{\frac{3n}{n+1}\right\}$ b) $\left\{\frac{5n^2}{n^3+1}\right\}$ c) $\left\{\frac{n^3+2}{n^2+n}\right\}$

a) Converges

$\lim_{n\to\infty} \frac{3n}{n+1} = 3$

think of Asymptotes

b) Converges

$\lim_{n\to\infty} \frac{5n^2}{n^3+1} = 0$.

Deg den. bigger

c) Diverges

Num top bigger

➤ **Arithmetic Sequence** ~ a sequence where $a_n = a_{n-1} + d$ where d common diff

➤ **Geometric Sequence** ~ a sequence where $a_n = a_{n-1} \cdot r$ where r common ratio

$\circ$ **Recursive** = in terms of previous term
$a_1 \quad a_n = a_{n-1}$

$\circ$ Explicit = $3n+1$

#21-24

Example

For each of the following arithmetic sequences, find (a) the common difference, (b) the tenth term, (c) a recursive rule for the nth term, and (d) an explicit rule for the nth term.

(1) $-7, -4, -1, 2, 5, \ldots$

(2) $\ln 3, \ln 9, \ln 27, \ln 81, \ldots$

① a) 3

b) $-7+3\cdot 9 = 20$ OR $3x-10$ (10-1)

Recursive c) $a_1 = -7$
$a_n = a_{n-1} + 3$ for $n \geq 2$

Exp. d) $a_n = -7 + 3(n-1)$
$-7+3n-3$
$= 3n-10$

② a) $\ln 9 - \ln 3 = \ln 9/3 = \ln 3$

b) $\ln 3 + (\ln 3)\cdot 9 = \ln(3\cdot 3^9)$
$9\ln 3$
$\ln 3^{10}$
$\boxed{\ln 59{,}049}$

c) $a_1 = \ln 3$
$a_n = a_{n-1} + \ln 3 \quad n \geq 2$

d) $a_n = \ln 3 + (n-1)\ln 3$
$\ln(3^1 \cdot 3^{n-1})$
$\boxed{\ln(3^n)}$

#25-28

Example

For each of the following geometric sequences, find (a) the common ratio, (b) the tenth term, (c) a recursive rule for the nth term, and (d) an explicit rule for the nth term.

(1) 2, 6, 18, 54, 162, ...

(2) $2^6, 2^9, 2^{12}, 2^{15}, 2^{18}, \ldots$

① a) 3

b) $2\cdot 3^9 = 39{,}366$

c) $a_1 = 2 \quad a_n = 3\cdot(a_{n-1}) \quad n \geq 2$

d) $a_n = 2\cdot 3^{n-1}$

② $\frac{2^9}{2^6}$

a) $2^3 = 8$

b) $2^6\cdot(2^3)^9$
$2^6\cdot 2^{27} = 2^{33}$
$\boxed{= 8{,}589{,}934{,}592}$

c) $a_1 = 2^6 \quad a_n = a_{n-1}\cdot 2^3 \quad a \geq 2$

d) $a_n = 2^6\cdot(2^3)^{n-1}$
$2^6\cdot 2^{3n-3}$
2^{3n+3}

#29-32 diff.

Example

The second and fifth terms of a sequence are 3 and 192, respectively. Find explicit and recursive formulas for the sequence if it is (a) arithmetic and (b) geometric.

a) $a_2 = a + d = 3$
$a_5 = a + 4d = 192$

$(a+4d) - (a+d) = 192 - 3$
$4d - d = 189$
$3d = 189$
$d = 63$

Recursive
$a_1 = -60$
$a_n = a_{n-1} + 63$

Exp.
$a_n = -60 + 63(n-1)$
$-60 + 63n - 63$
$63n - 123$

b) $a_2 = a\cdot r = 3$
$a_5 = a\cdot r^4 = 192$
$\frac{a\cdot r^4}{a\cdot r} = \frac{192}{3}$
$r^3 = 64$
$r = 4$

Recursive
$a_1 = 3/4$
$a_n = a_{n-1}\cdot 4$

Exp
$a_n = 3/4\cdot(4)^{n-1}$

#33-36 **Example**

Graph the sequence $b_n = \sqrt{n} - 3$

-mode Sequence

- Y= n min 1
$u(n) = \sqrt{n} - 3$
$u(n\text{min}) = 0$

Graph

-Use trace for pts

① mode Seq.

② 2nd, Stat, Ops, seq.
{Eqn, var, start, end}

- **Fibonacci Sequence**

1

$a_0 = 1$

$a_1 = 1 + 1 = 2$

$a_3 = 3$

5

8

13

9.5 Series

- **Summation Notation** Sum of a sequence $\{a_1, a_2 + \ldots a_n\}$

$$\sum_{k=1}^{n} a_k$$

sum, n ← end, k=1 ↑ start

Ex 2, 5, 8, 11, ... 29

$$\sum_{k=1}^{10} (3k-1)$$

Find the number represented by each of the following expressions.

① $\sum_{k=1}^{5} 3k$ ② $\sum_{k=5}^{8} k^2$ ③ $\sum_{n=0}^{12} \cos(n\pi)$

④ $\sum_{n=1}^{\cancel{8}5} \sin(n\pi)$ ⑤ $\sum_{k=1}^{\cancel{8}2} \frac{3}{10^k}$

① 3, 6, 9, 12, 15 = 45

② $5^2, 6^2, 7^2, 8^2 = 174$

③ ① sum(seq(cos(nπ)), n, 0, 12) = 1

Formula ↑ ; ↑ start ; ↑ end

④ -2.10^{-31} ≈ 0

2nd Stat math sum ; Stat math ops

⑤ $\frac{3}{10} + \frac{3}{100}$ .33

Example

A corner section of a stadium has 10 seats along the front row. Each successive row has 5 more seats than the row preceding it. If the top row has 70 seats, how many seats are in the entire section?

$a_1 = 10$

$a_n = 70$

$d = 5$

$a_n = a_1 + d(n-1)$

$70 = 10 + 5(n-1)$

$70 = 10 + 5n - 5$

$70 = 5 + 5n$

$65 = 5n$

$n = 13$ rows

$n\left(\frac{a_1 + a_n}{2}\right)$

$13\left(\frac{10+70}{2}\right)$

$= 520$

sum(seq 5+5x, x, 1, 13, 1)

Sum of Arithmetic Sequence	Sum of Geometric Sequence
$n\left(\frac{a_1+a_n}{2}\right)$ $=\frac{n}{2}\left(2a_1+(n-1)d\right)$	$\frac{a_1(1-r^n)}{1-r}$ Proof pg 680

#1-12

Example

#13-16

Find the sum of the geometric sequence 9, 9/7, 9/49, 9/343, ..., $9(1/7)^7$.

$a_1=9$

$r=1/7$

$n=9(1/7)^{7}$ $\leftarrow n-1$ so $n=8$

$$\frac{9\left(1-(1/7)^8\right)}{1-1/7}=10.49$$

sum(seq(9 · 1/7^{n-1}, n, 1, 8))

Example

For each of the following series, find the first five terms in the sequence of partial sums. Which of the series appear to converge?

.1 .11 .111

a) 0.1 + 0.01 + 0.001 + 0.0001 + …… Converges 1/9

b) 10 + 20 + 30 + 40 + …… 10, 30, 60, 100 Diverges

c) 1 -1 + 1 – 1 + … 1, 0, 1, 0 Diverges

Geo Converge iff $|r|<1$ & converges to $a/(1-r)$ Proof 683

Example

Determine whether the series converges. If it converges, give the sum.

(a) $\sum_{k=1}^{\infty} 6(0.25)^{k-1}$

(b) $\sum_{n=0}^{\infty} \left(\frac{-12}{33}\right)^n$

(c) $\sum_{n=1}^{\infty} \left(\frac{17}{16}\right)^n$

(d) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$

a) $r=.25$ $|r|<1$ Converge

$6/(1-.25)=8$

b) $r=-12/33$ $\frac{12}{33}<1$ Converges

$1/1-12/33 = 11/15$

0.733

c) $r=17/16$ $\frac{17}{16}>1$ Diverges

d) $a=1$

$r=1/3$

$1/3<1$ Converges

$\frac{1}{1-1/3}$ 1.5

9.6 Mathematical Induction

Principle of Mathematical Induction

Let P_n be a statement about integers n. then P_n is true for all positive integers n provided the following conditions are satisfied:

1) (the anchor) P_1 is true Show
2) (the inductive step) if P_k is true, then P_{K+1} is true.

Assume

Example

Prove that $1+2+3+...+n=\frac{n^2+n}{2}$ is true for all positive integers n.

Anchor: $n=1$ P_1 $1=\frac{1^2+1}{2}$

$1=1$

- Assume true for P_K

So $1+2+3+\ldots+K=\frac{K^2+K}{2}$

- Add $K+1$ to both sides

$1+2+3+\ldots+K+K+1=\frac{K^2+K}{2}+(K+1)\frac{2}{2}$

$=\frac{K^2+K}{2}+\frac{2K+2}{2}$

$=\frac{K^2+K+2K+2}{2}$

$=\frac{K^2+2K+1+K+1}{2}$

$=\frac{(K+1)^2+(K+1)}{2}$

$\therefore P_{K+1}$ is true

We want to show $\frac{(K+1)^2+(K+1)}{2}$

K^2+2K+1

Example

$1^3 \quad 2^3 \quad 3^3 \qquad (K+1)^3$

Prove that $1+8+27+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$ is true for all positive integers n.

- Anchor P_1 $1^3 = \frac{1^2 \cdot (1+1)^2}{4}$

$1^3 = 4/4 \quad 1=1$

- Hyp: P_K is true $1+8+27+\ldots+K^3 = \frac{K^2(K+1)^2}{4}$

- Ind: P_{K+1} Add $(K+1)^3$ to both sides

$1+8+27+\ldots+K^3+(K+1)^3 = \frac{K^2(K+1)^2}{4} + (K+1)^3 \frac{4}{4}$

K^2+2K+1

$= \frac{K^2(K+1)^2}{4} + \frac{4(K^3+3K^2+3K+1)}{4}$

$= \frac{K^4+2K^3+K^2+4K^3+12K^2+12K+4}{4}$

$K^2(K^2+4K+4)+2K(K^2+4K+4)+(K^2+4K+4)$

$\frac{(K^2+4K+4)(K^2+2K+1)}{4}$

$\boxed{\frac{(K+1)^2(K+2)^2}{4}}$

We want $(K+1)$ RHS

$\frac{(K+1)^2(K+2)^2}{4}$

$K^2+2K+1 \qquad K^2+4K+4$

Example

Prove that $5^n - 1$ is evenly divisible by 2 for all positive integers n.

Anchor $P_1 = 5^1-1 = 4$ Div by 2

- Hyp P_K true so 5^K-1 div by 2

- Next is $K+1$, Show Div by 2

$5^{K+1}-1$

$5^K \cdot 5^1 - 1$

$5 \cdot 5^K - 1$

$5(5^K-1)+4$

Result of 2 ... by 2

$a^{3+2} = a^3 \cdot a^2$

9.7 Statistics and Data

- **Categorical Variable** – non-numeric
  "Qual." Shoes, color, gender
- **Quantitative Variable** – variable that is numeric
- **Stemplots** – Table

Stem	Leaf
2	3 5 7
3	2 4

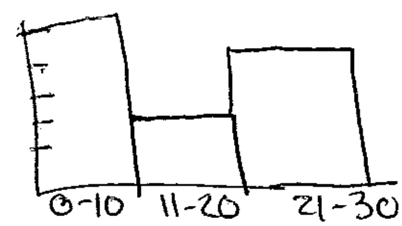
23, 25, 27
32, 34

- **Frequency Tables** →

Int.		Freq.
0-10	~~||||~~	5
11-20	||	2
21-30	||||	4

- **Histograms**

Freq.



- **TimePlots** – line graph w/ time

Example

Twenty-six people tried this activity. At the end of what each person judged to be a minute, the actual time that had elapsed was recorded to the nearest second. The responses (in seconds) were as follows:

26 Responses to the Question			
63	67	79	75
57	72	~~52~~	89
~~39~~	59	55	68
66	86	70	~~52~~
60	64	~~42~~	~~54~~
56	82	57	65
59	~~33~~		

Stem	Leaf
3	3 9
4	2
5	2 2 4 5 6 7 7 9 9
6	0 3 4 5 6 7 8
7	0 2 5 9
8	2 6 9

Make a stemplot for the data.

$K + a\sin(b(t-h))$

Example

Mark McGwire and Barry Bonds entered the major leagues in 1986 and had overlapping careers until 2001, the year that McGwire retired. During that period they averaged 101.63 and 144.56 hits per year, respectively. Compare their annual hit totals with a back-to-back stemplot. Can you tell which player was a more consistent hitter?

Major League Hit Totals for Mark McGwire and Barry Bonds through 2001

Year	86	87	88	89	90	91	92	93	94	95	96	97	98	99	00	01
McGwire	10	161	143	113	123	97	125	28	34	87	132	148	152	145	72	56
Bonds	92	144	152	144	156	149	147	181	122	149	159	155	167	93	147	156

McGuire		Bonds
0	1	
8	2	
4	3	
	4	
6	5	
	6	
2	7	
7	8	
7	9	2 3
	10	
3	11	
5 3	12	2
2	13	
8 5 3	14	4 4 7 7 9 9
2	15	2 5 6 6 9
1	16	7
	17	
	18	1

Bonds

Example

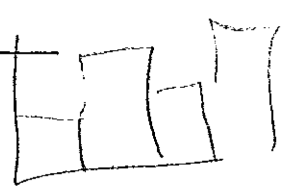
Make a histogram of Mickey Mantle's annual home run totals given in the table below, using intervals of width 5.

Annual Home Run Statistics For Mickey Mantle

Year	Home Runs	Year	Home Runs	Year	Home Runs
1951	13	1957	34	1963	15
1952	23	1958	42	1964	35
1953	21	1959	31	1965	19
1954	27	1960	40	1966	23
1955	37	1961	54	1967	22
1956	52	1962	30	1968	18

Source: www.baseball-reference.com

1 Enter Data L_1
2 Stat Plot ON
3 window [0, 60] x
\# of HR
xscl 5 (intervals)
[-1, 5] ← freq.
y



Example

The table below gives the average monthly temperature of Albany, New York for the past 50 years. Display the data in a time plot and analyze the 12-month trend.

1 Enter L_1 + L_2
2 [9, 13] [0, 80]
3 Stat Plot (2nd y=)

Average Monthly Temperatures of Albany, New York for the Past 50 Years

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
°F	22	24	34	47	58	67	72	70	61	51	40	27

9.8 Statistics and Data (Algebraic)

- **Parameter** – describes an entire population
- **Inferential Statistics** – sample
- **Mean** – avg
- **Median** – middle high to low
- **Mode** – often
- **Five Number Summary**

outliers

category

0 80 90 95 (85)

66.25

{min, FQ, median, TQ, max}

Q_1 LQ Q_3 UQ

Example

A teacher gives a 10-point quiz and records the scores in a frequency table as shown below. Find the mode, median, and mean of the data.

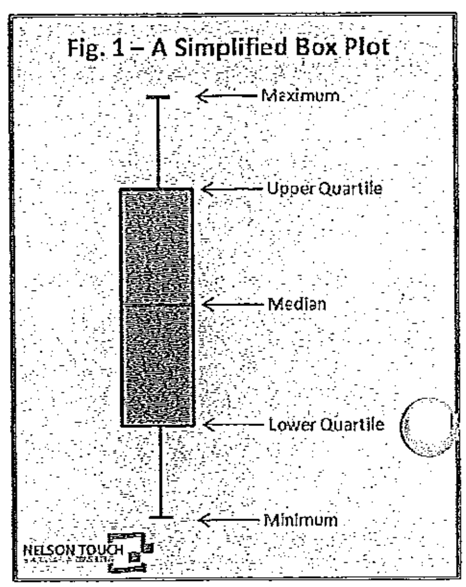
Quiz Scores											
Score	10	9	8	7	6	5	4	3	2	1	0
Frequency	4	4	4	4	2	2	5	3	0	2	2

= 32 students (16)

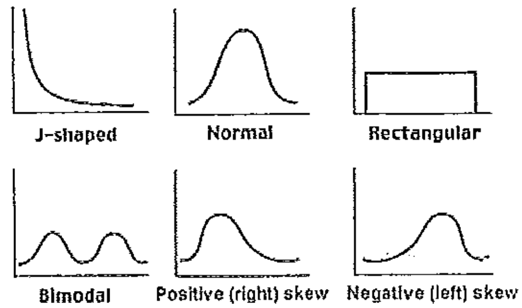
mode 4

median 6.5

mean 5.9



- **Shape of Distribution**
  - **Symmetric**
  - **Skewed Right**
  - **Skewed Left**



>Outlier = # more than 1.5 * IQR $Q_1 - 1.5IQR$
InterQuart. Range $Q_3 - Q_1$ $Q_3 + 1.5IQR$ 172.25

Example

Draw boxplots for the male and female data given below and describe briefly the information displayed in the visualization.

Male and Female Cancer Death Rates Per 100,000 Population Among 12 Countries from 1986–88

Country	Male	Female
Hungary	~~235.4~~	129.4
Italy	~~193.7~~	99.7
United States	~~163.2~~	109.7
Australia	~~162.8~~	102.0
Cuba	~~129.0~~	95.9
Japan	~~149.8~~	78.1
Canada	~~170.6~~	111.5
France	~~204.7~~	89.9
Scotland	~~200.9~~	136.1
England & Wales	~~181.9~~	126.9
Switzerland	~~173.9~~	99.5
Ecuador	~~83.9~~	86.1

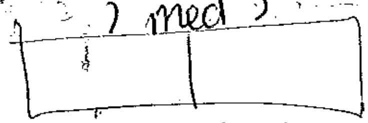
Source: World Health Organization Data as adapted by the American Cancer Society, 1992

Males

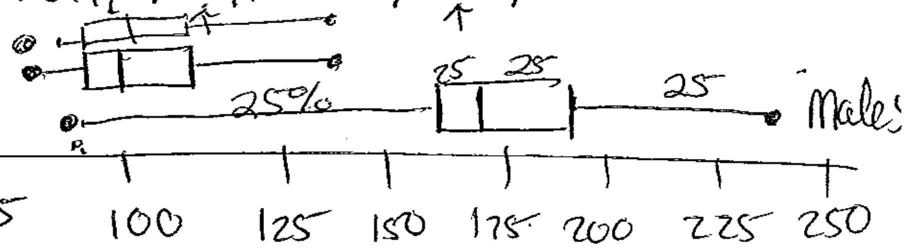
83.9 ~~129~~ ~~149.8~~ ~~162.8~~ ~~163.2~~ ~~170.6~~ ~~173.9~~

~~181.9~~ 193.7 200.9 ~~204.7~~ ~~235.4~~

{83.9, 156.3, 172.25, 197.3, 235.4} males
outlier   med



{78.1, 92.9, 100.85, 119.2, 136.1} females



- **Standard Deviation** – measure of how data set is spread.

$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x-\bar{x})^2} \longrightarrow \sigma$ "Sigma" $\sqrt{\text{variance}}$

>Variance = avg squared diff. from mean

Ex
Standard dev. 600, 470, 170, 430, 300

① mean = 394

② Variance $\dfrac{206^2 + 76^2 + (-224)^2 + 36^2 + (-94)^2}{5} = 21{,}704$

③ $\sqrt{21{,}704} \approx 147$ σx

Example

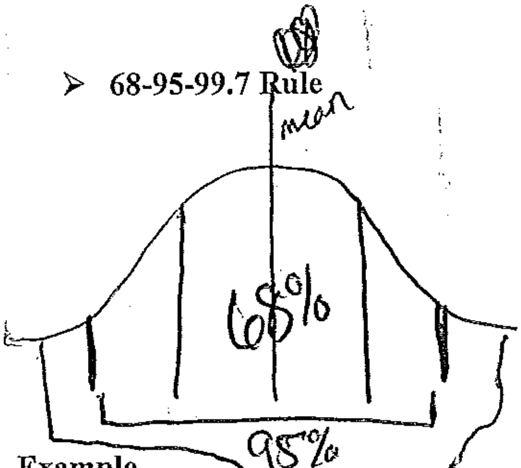
Based on the research data presented in Alternate Example 10, would a loon chick weighing 86.2 grams be in the top 2.5% of all newly hatched loon chicks?

mean 81.63
Stand dev. 2.23

Day 2

#42

➢ 68-95-99.7 Rule

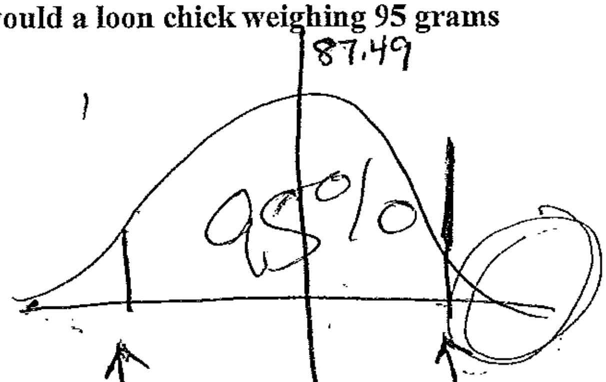


Normal Dist. mean $\bar{x}$ μ
Stand dev. σ

68% data $\mu - 1\sigma$ $\mu + 1\sigma$
95% data $\mu - 2\sigma$ $\mu + 2\sigma$
99.7% data $\mu - 3\sigma$ $\mu + 3\sigma$

Example

Based on the research data in example 10 (pg. 711), would a loon chick weighing 95 grams be in the top 2.5% of all newly hatched loon chicks?



Mean = 87.49
Stdev. = 3.51

YES

87.49 − 2*3.51 = 80.47

87.49 + 2*3.51
94.51

2001 Mean 21
Standev. 4.7

2012 Mean 20.1

>ACT Avg 20.8

9.9 Statistical Literacy

- **Positive Association**
- **Negative Association**
- **CORRELATION DOES NOT IMPLY CAUSATION!!!**
- **Bias**

① Undercoverage
② Voluntary Response
③ Response bias = due to wording

- **Experimental Design**
  - **Control**
  - **Randomization** RandInt(
  - **Replication**
  - **Blocking** – making pre-existing cond. spread out evenly. Q
- observational studies – passively watch & record

1-39 odd

1.