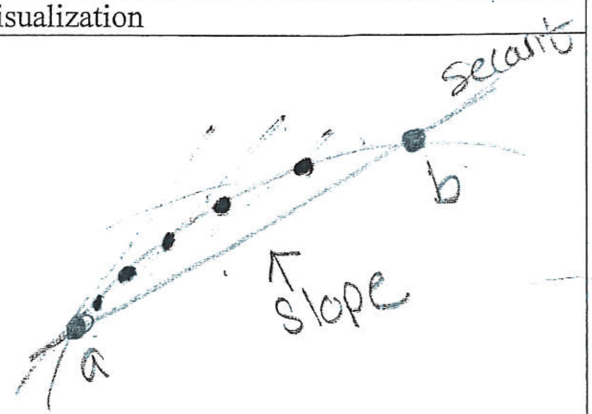
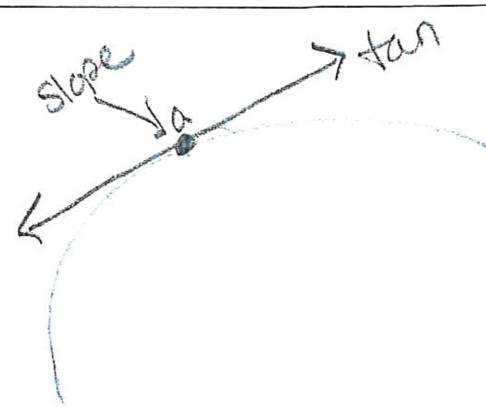
10.1 Limits and Motion: The Tangent Problem

- Speed – magnitude of velocity (Pos. or Zero)

$\frac{\Delta s}{\Delta t}$

- Velocity – ratio of chng is position to chng in time (Pos, Neg, Zero)

	Definition	Visualization
Average	$\frac{s(a)-s(b)}{a-b}$	
Instantaneous	specific pt in time. $t = a$ $\lim_{h \to 0} \frac{s(a+h)-s(a)}{h}$ $a+h \approx a$	

Example

#1-2

An airplane travels 1500 miles in 3 hours and 45 minutes. What is the average velocity of the plane over the entire 3.75-hour time interval?

$$\frac{\Delta s}{\Delta t} = \frac{1500 \text{ miles}}{3.75 \text{ hr}} = \boxed{400 \text{ mph}}$$

Example

In a time of t seconds, a particle moves a distance of s meters from its starting point where $s = 3t^2$.

a) Find the average velocity between t = 1 and t = 1 + h if:

(i) h = 0.1, (ii) h = 0.01, (iii) h = 0.001

$\frac{\Delta s}{\Delta t}$

$\frac{3(1.1)^2 - 3(1)^2}{1.1 - 1}$ = 6.3

$\frac{3(1.01)^2 - 3(1)^2}{1.01 - 1}$ = 6.03

$\frac{3(1.001)^2 - 3 \cdot 1^2}{1.001 - 1}$ = 6.003

b) Use your answers to part a to estimate the instantaneous velocity of the particle at time t = 1.

6

Can't actually be 1 b/c then dividing by zero!

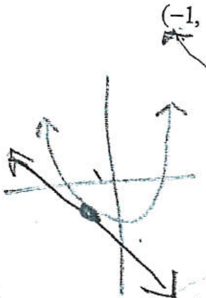
- Tangent Line – line representing instanteous velocity at a pt. rate of chg.

#3-5!

Example

Use limits to find the slope of the tangent line to the graph ... (−1, −3).

equation of line ... $-2(x - -1)$



Neg Slope −2

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\frac{S(-1+h) - S(-1)}{-1 + h - -1}$

$\frac{(-1+h)^2 - 4 - ((-1)^2 - 4)}{h}$

$\frac{1 - 2h + h^2 - 4 + 3}{h}$

$\frac{-2h + h^2}{h}$

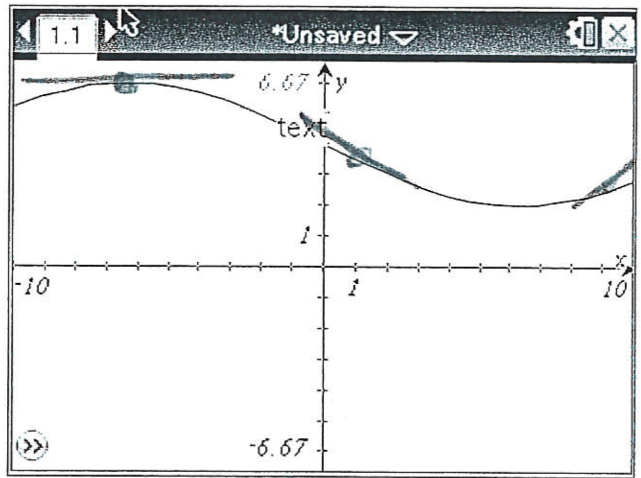
$\lim_{t \to} \frac{\cancel{h}(-2+h)}{\cancel{h}}$

$= -2$

- The Derivative

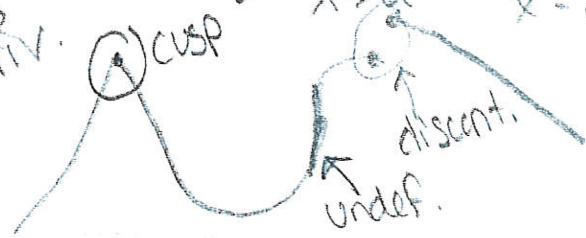
$f'(a)$ $\frac{dy}{dx}$

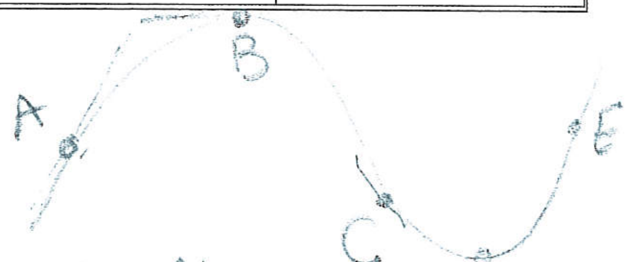
"Prime"



$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

No deriv.





Slope At +, −, 0

A +

B 0

C Neg

D 0

E Pos

#15-22 **Example**

Find $f'(-1)$ if $f(x) = 3x^2 + 2$.

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \frac{3(-1+h)^2 + 2 - (3(-1)^2 + 2)}{h}$$

$$= 3(1 - 2h + h^2) + 2 - 5 \;/ h$$

$$= \cancel{3} - 6h + 3h^2 - \cancel{3} \;/ h$$

$$= \frac{-6h + 3h^2}{h}$$

stop

$$= \frac{\cancel{h}(-6 + 3h)}{\cancel{h}}$$

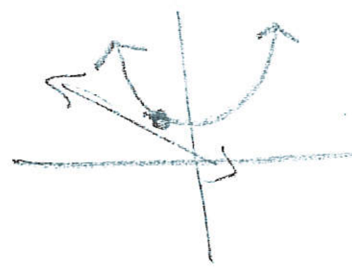
$$= -6 + 3h$$

$$\boxed{= -6}$$

Example

(a) Find $f'(x)$ if $f(x) = \sqrt{x}$.

(b) Find $\frac{dy}{dx}$ if $y = \frac{2}{x+5}$.



$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

Rationalize num.

$$= \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\boxed{= \frac{1}{2\sqrt{x}}}$$

$\frac{2}{x+5}$

$$\frac{f(x+h)-f(x)}{h}$$

$$= \frac{\frac{2}{x+h+5} - \frac{2}{x+5}}{h}$$

$$\frac{\frac{2(x+5)-2(x+h+5)}{(x+h+5)(x+5)}}{h}$$

$$= \frac{2x+10-2x-2h-10}{(x+h+5)(x+5)}$$

$$\frac{\frac{-2h}{(x+h+5)(x+5)}}{h}$$

$$\frac{-2h}{(x+h+5)(x+5)} \cdot \frac{1}{h}$$

$$= \frac{-2}{(x+h+5)(x+5)}$$

$$= \frac{-2}{(x+5)(x+5)}$$

$$= \boxed{\frac{-2}{(x+5)^2}}$$

Example

1. Find the derivative algebraically for the following:

$f(x) = 2x^2 - 3$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \frac{2(x+h)^2-3 - (2x^2-3)}{h}$$

$$= \frac{2(x^2+2xh+h^2)-3-2x^2+3}{h}$$

$$= \frac{\cancel{2x^2}+4xh+2h^2\cancel{-3}-2x^2+3}{h}$$

$$\frac{4xh+2h^2}{h}$$

$$= \frac{\cancel{h}(4x+2h)}{\cancel{h}}$$

$$= 4x$$

---

2. $f(x) = \dfrac{1}{(x+1)}$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

$$= \frac{\frac{x+1-(x+h+1)}{(x+h+1)(x+1)}}{h}$$

$$\frac{\frac{-h}{(x+1)(x+h+1)}}{h}$$

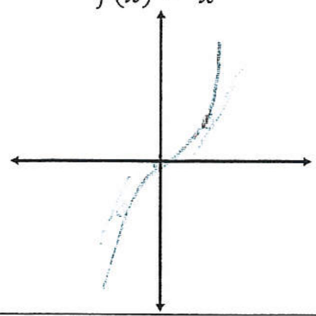
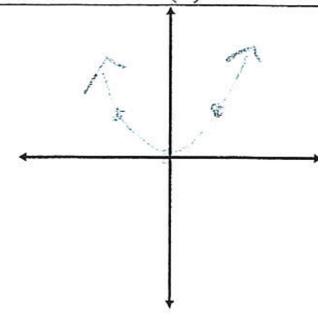
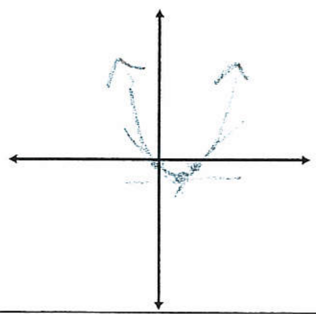
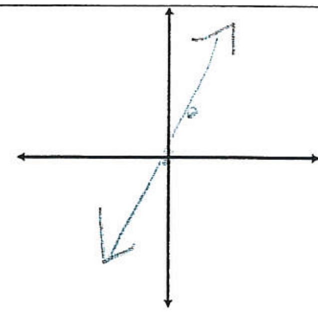
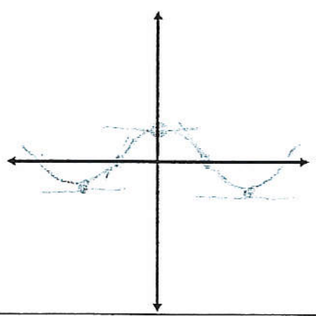
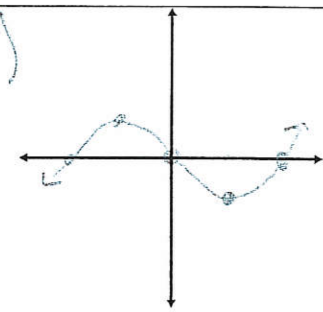
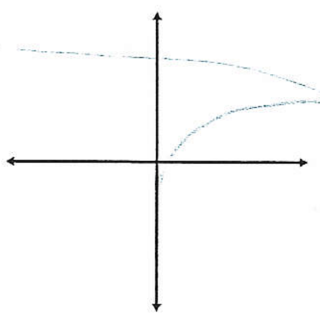
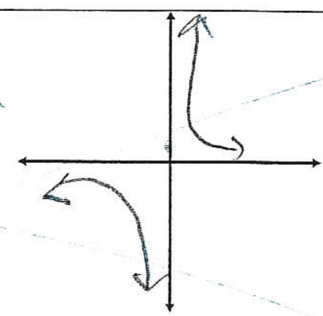
$$\frac{-\cancel{h}}{(x+1)(x+h+1)} \cdot \frac{1}{\cancel{h}}$$

$$= \frac{-1}{(x+1)(x+h+1)}$$

$$= \frac{-1}{(x+1)^2}$$

Example

Sketch the graph for the following functions and use these to sketch the graph of $f'(x)$.

F(x)	F'(x)
$f(x) = x^3$ 	$3x^2$ 
$f(x) = x(x-1) = x^2 - 1$ 	$2x$ $(5,0)$ 
$f(x) = \cos(x)$ 	$-\sin(x)$ 
$f(x) = \log x$ 	$= \frac{1}{x \ln a}$ $1/x$ 

10.2 Limits and Motion: The Area Problem

Recall: Distance = Rate X Time

Example

#1-6

An automobile traveled at a constant rate of 39 mph for 2 hours and 30 minutes. How far does the automobile travel?

$39 \cdot 2.5 = \boxed{97.5 \text{ miles}}$

Example

An automobile travels at an *average* rate of 54 miles per hour for 3 hours and 30 minutes. How far does the automobile travel?

$54 \cdot 3.5 = \boxed{189 \text{ miles}}$

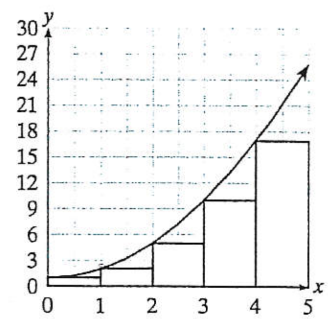
Example

#11-16

17-20 a

Use the five rectangles in the figure to estimate the area of the region below the curve $f(x) = x^2 + 1$ for x in the interval [0, 5].

LRAM



Rect. area = b·h

Subinterval	Base of rect	Ht of rect.	Area
[0, 1]	1	$f(0) = 0^2 + 1 = 1$	$1 \cdot 1 = 1$
[1, 2]	1	$f(1) = 2$	2
[2, 3]	1	$f(2) = 5$	5
[3, 4]	1	$f(3) = 10$	10
[4, 5]	1	$f(4) = 17$	17

Total $35u^2$

under estimate

(13) [0, 2]   4   2/4   ~~Ex~~ [3, 6]   12 int

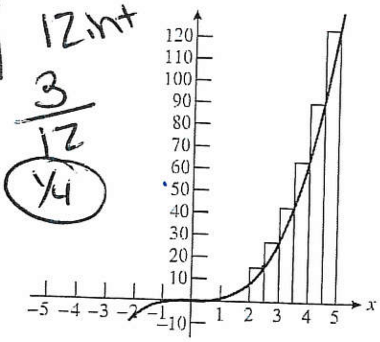
$\frac{3}{12}$

1/4

[0, .5]

[.5, 1]

exam



Example

17-20b

Use the six rectangles in Figure 10.2 to approximate the area of the region below the graph of $f(x) = x^3$ over the interval [2, 5].

Int	Base	Ht of Rect	Area
[2, 2.5]	1/2	f(2.5) = 15.625	7.8125
[2.5, 3]	1/2	27	13.5
[3, 3.5]	1/2	42.875	21.4375
[3.5, 4]	1/2	64	32
[4, 4.5]	1/2	91.125	45.5625
[4.5, 5]	1/2	125	62.5
			182.8125

over estimate

Stop

➢ **Definite Integral**

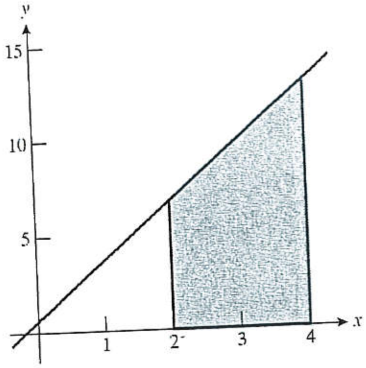
Let f be a function on $[a, b]$ and let $\sum_{i=1}^{n} f(x_i)\Delta x$ be defined as above. The definite integral of f over $[a, b]$, denoted $\int_a^b f(x)dx$, is given by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{a} f(x_i)\Delta x,$$

provided the limit exists.

If the limit exists, we say f is integrable on $[a, b]$.

Example



Find $\int_2^4 3x+1\,dx$.

Note trapezoid so

$$A = h\left(\frac{b_1+b_2}{2}\right)$$

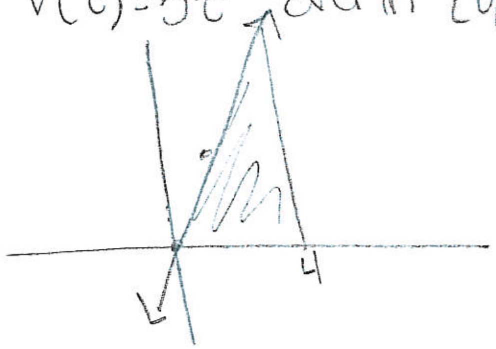
$$= 2\left(\frac{3(2)+1) + 3(4)+1}{2}\right)$$

$$= 2\left(\frac{7+13}{2}\right) = 20$$

Example

Suppose a ball rolls down a ramp so that its velocity after t seconds is always $5t$ feet per second. How far does it fall during the first 4 seconds?

$v(t) = 5t$ over in $[0,4]$



Δ so

$\frac{1}{2}bh$

$\frac{1}{2}\cdot 4\cdot 5(4)$

40

① $\int_2^5 3\,dx$

② $\int_{-3}^{3} \sqrt{9-x^2}$ semicircle $\frac{1}{2}\pi r^2$

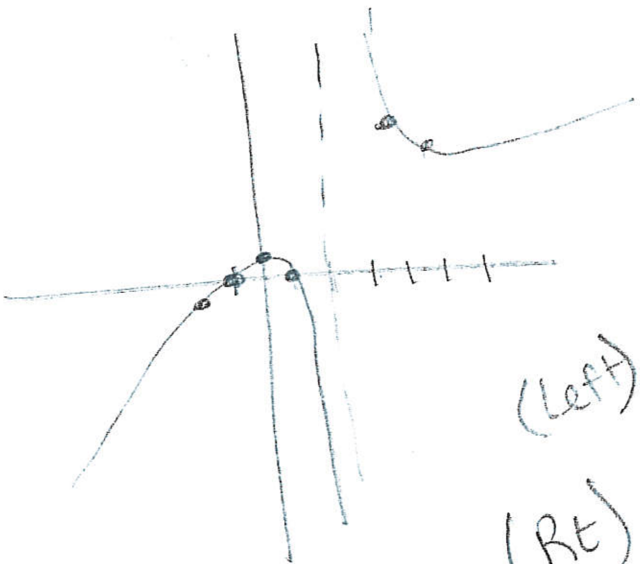
10.3 More on Limits

Recall: Definition of Limit at a $\lim_{x\to a} f(x) = L$

$f(x)$ gets close to L as x gets close to a.

Example

Find $\lim_{x\to 2} \frac{x^2-1}{x-2}$



2 is an asymptote

(Left) $\lim_{x\to 2^-} \frac{x^2-1}{x-2} = -\infty$

(Rt) $\lim_{x\to 2^+} \frac{x^2-1}{x-2} = \infty$

Properties of Limits

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then

1. **Sum Rule** $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. **Difference Rule** $\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. **Product Rule** $\lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
4. **Constant Multiple Rule** $\lim_{x\to c}(k\cdot g(x)) = k\cdot \lim_{x\to c} g(x)$
5. **Quotient Rule** $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$
6. **Power Rule** $\lim_{x\to c}(f(x))^n = (\lim_{x\to c} f(x))^n$ for n a positive integer
7. **Root Rule** $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ for $n \geq 2$ a positive integer, provided $\sqrt[n]{\lim_{x\to c} f(x)}$ and $\lim_{x\to c}\sqrt[n]{f(x)}$ are real numbers.

Recall

$\lim_{x\to 1} \frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1}$

Synth.

$$\begin{array}{r|l} & x^2+1 \\ \hline x-1 & x^3-1 \\ & -(x^3-1x) \\ \hline & 1x-1 \\ & -1(x-1) \\ \hline & 0 \end{array}$$

$$\frac{(x-1)(x+3)}{x-1} \quad \lim_{x\to 1} \frac{x^2+2x-3}{x-1}$$

#1-10

Example

Use limit properties to justify $\lim_{x\to 3} \frac{x^2+5x}{x+9} = \frac{3^2+(5)(3)}{3+9} = 2$

Can use direct sub.

$$\lim_{x\to 3} \frac{x^2+5x}{x+9}$$

$$= \frac{\lim_{x\to 3} x^2+5x}{\lim_{x\to 3} x+9} \quad \text{(Prop 5)}$$

$$= \frac{\lim_{x\to 3} x^2 + \lim_{x\to 3} 5x}{\lim_{x\to 3} x + \lim_{x\to 3} 9} \quad \text{(Prop 1)}$$

$$= \frac{\left(\lim_{x\to 3} x\right)^2 + 5\left(\lim_{x\to 3} x\right)}{\lim_{x\to 3} x + \lim_{x\to 3} 9} \quad (3\,\&\,4)$$

$$= \frac{3^2+5\cdot 3}{3+9} \quad \text{Sub}$$

$$2$$

Example

Find each of the following limits:

(a) $\lim_{x\to 0} \frac{\sin 3x + 2x}{5}$ (b) $\lim_{x\to 1/2} \frac{\ln 2x + \cos \pi x}{x - \sin \pi x}$

#11-26

a) $\lim_{x\to 0}\left(\frac{\sin 3x}{5} + \frac{2x}{5}\right)$ Sum

$= \lim_{x\to 0} \frac{\sin 3x}{5} + \lim_{x\to 0} \frac{2x}{5}$

$= 0 + 0$

$= 0$

stop

b) $\frac{\lim_{x\to 1/2} \ln 2x + \lim_{x\to 1/2} \cos \pi x}{\lim_{x\to 1/2} x - \lim_{x\to 1/2} \sin \pi x}$

$= \frac{\ln(2\cdot .5) + \cos \frac{1}{2}\pi}{\frac{1}{2} - \sin \frac{1}{2}\pi}$

$= \frac{0+0}{\frac{1}{2}-1} = 0$

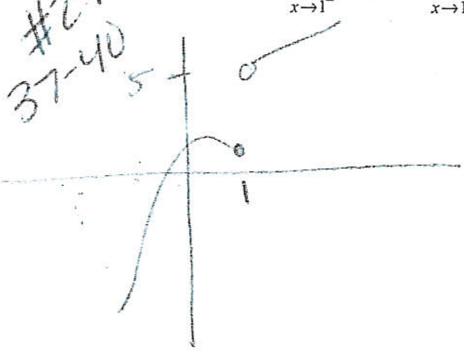
Stop

#27-32

37-40

Example

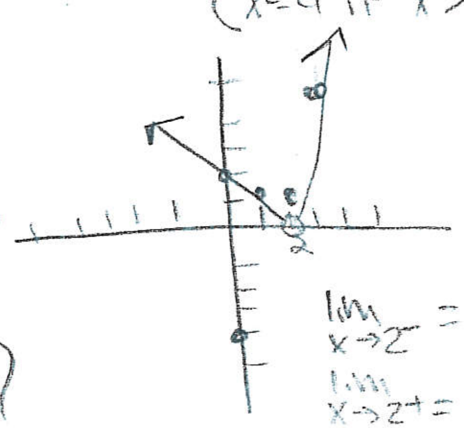
Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$ where $f(x) = \begin{cases} x^3 - x + 1 & \text{if } x \le 1 \\ 5x+1 & \text{if } x > 1 \end{cases}$



$\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1^+} f(x) = 6$

Ex #37 $a=2$ $\begin{cases} 2-x & \text{if } x<2 \\ 1 & \text{if } x=2 \\ x^2-4 & \text{if } x>2 \end{cases}$



$\lim_{x\to 2^-} = 0$

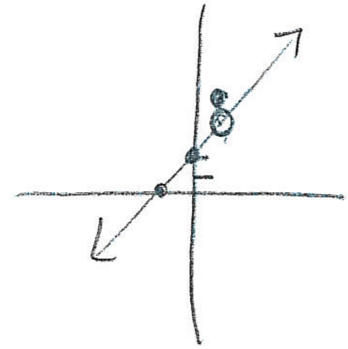
$\lim_{x\to 2^+} = 0$

$\lim_{x\to 2} = 0$

Example

Find lim f(x) and prove that f is discontinuous at x = 2.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} \ (x+2) & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$



$$\lim_{x\to 2} \frac{(x+2)(x-2)}{x-2}$$

$$\lim_{x\to 2} = x+2$$

$= 4$ but $f(2) = 5$ so discontinuous

Example

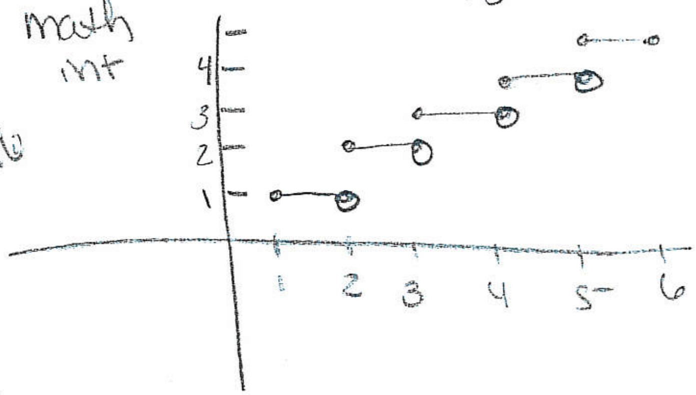
Let f(x) = int(x). Find a) $\lim_{x\to 5^-} \text{int}(x)$ 4 b) $\lim_{x\to 5^+} \text{int}(x)$ 5 c) $\lim_{x\to 5} \text{int}(x)$

DNE

b/c diff.

math int

#41-46



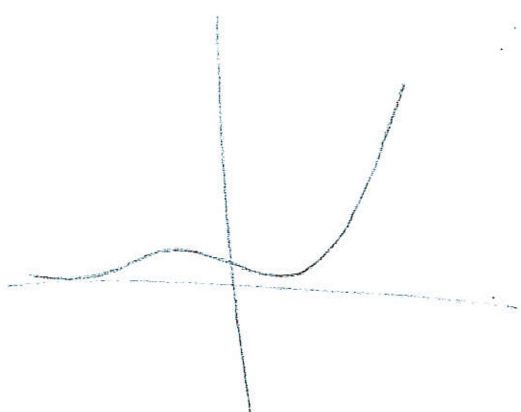
Step

Example

Let $f(x) = x^2 e^x$. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$.

Table + Graph

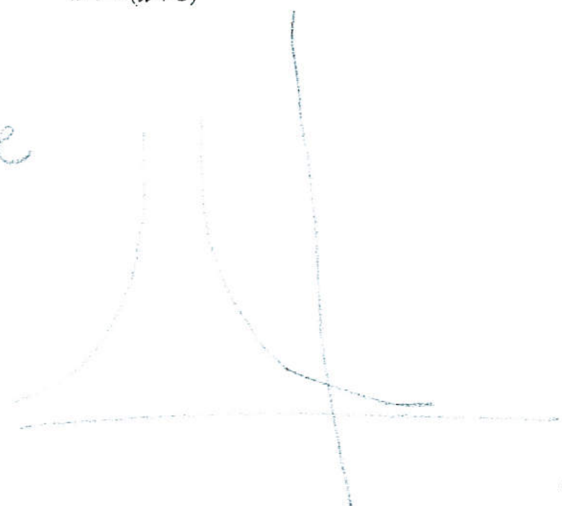
$\uparrow$ ∞ $\quad$ $\uparrow$ 0



Example

Find $\lim\limits_{x \to -5} \dfrac{3}{(x+5)^2}$.

Graph & Table



$\lim\limits_{x \to -5^+} = \infty$ $\lim\limits_{x \to -5^-} = \infty$

∞

So does Not exist

Vertical Asymptote $x = -5$

zeros of Denom

Horiz. Asymptote

Math nDeriv(funct, x, 3 pt) 8K NDER

10.4 Numerical Derivatives and Integrals

at pt 3

Examples
Find the derivatives of

a) 5x^2 + 3x + 2 $\boxed{10x+3}$ 33

b) 0.1x^2 + 2 x 0.1x^3 + 2$x^{\sqrt{2}}$ (.2x³) .2x + .6x² + 2√2 x^(√2−1) 10.45

c) $\sqrt[4]{x}$ $x^{1/4}$ $\frac{1}{4}x^{-3/4}$ .1097

Ex Integrals NINT

$\int_0^5 x+3$ 27.5 (x+3, x, 0, 5) ← 4 Graph

$\int_2^7 x^2$ 111.6

Example

An automobile is driven at a variable rate along a test track for 4 hours so that its velocity at any time t $(0 \le t \le 4)$ is given by $v(t) = 20 + 5\cos 3t$ miles per hour. How far does the automobile travel during the 4-hour test?

$\int_0^4 (20+5\cos 3t)$

nInt(20+5cos 3x, x, 0, 4)

= 79.11 miles

10.4 Numerical Derivatives and Integrals

Examples

Find the derivatives of

a) 5x^2 + 3x + 2   b) 0.1x^2 + 2 x 0.1$x^3 + 2x^{\sqrt{2}}$ c) $\sqrt[4]{x}$

Example

#3